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Mathematical structures in field theories

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Preface

This volume covers the greater part of the lectures presented in the seminar "Mathematical Structures in Field Theories" during the academic year 1986-1987.

The contributions are of a somewhat diverse nature and we have ordered them accordingly. The first three chapters contain the lectures in which the emphasis lies on the mathematics. They are however of great importance for a sound foundation of the physical formalism. Chapter 1 contains a mathematical interpretation of Dirac's formalism. Chapter 2, 3 and 4 are all related to supersymmetric field theory and bear a more direct relationship to physics, although here the mathematical features are also manifestly present.

The organizers of the seminar want to express their acknowledgement to the authors who contributed to this volume as well as to the people at the Centre for Mathematics and Computer Science who managed in transforming the contributions into high standard typesetting and printing.

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Dirac's Formalism and Functional Analysis

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INTRODUCTION

In his 'Principles of quantum mechanics' Dirac approaches quantum mechanics by means of a symbolic method, the so-called bracket formalism. This formalism has a mathematical flavour, but, in fact, is based upon bold claims which lack mathematical foundation. In the preface to the first edition of his celebrated monograph Dirac explains why he is in favour of this symbolic method. He observes (and here we quote) 'The symbolic method, however, seems to go more deeply into the nature of things. It enables one to express the physical laws in a neat and concise way and will probably be increasingly used in the future as it becomes better understood and its own special mathematics gets developed'. Mathematicians have been searching both for a pure mathematical description of quantum mechanics and for a mathematical basis for Dirac's bracket formalism. We mention John von Neumann, the founding father of Hilbert space theory and Laurent Schwartz, the founding father of distribution theory. However, Hilbert space theory is too limited to fulfill the needs of quantum mechanics. Also, Schwartz distribution theory does not give all satisfactory solutions although people ungroundedly believe so.

Many mathematicians have felt inspired by Dirac's work. Yet, nobody succeeded in developing a theory which provides a mathematical interpretation of *all* aspects of Dirac's formalism and probably nobody ever will.

In this series of lectures we present a mathematical interpretation of the bracket formalism which is more in line with Dirac's original ideas than any interpretation we know of. We illustrate the latter claim with the following example.

In Dirac's formalism two kind of vector spaces appear, the space B of bra vectors and the space K of ket vectors. Dirac assumes a sesquilinear form, the bracket, on the space $K \times B$ and, also, that each ket is in one-one correspondence with a bra. Mathematically this means that Dirac assumes the existence of a pairing between generalized functions. It has been shown that such a pairing does not exist.

In almost all mathematical interpretations one feigns that these claims of Dirac are not there and one represents the bra space by a space of test functions and the ket space by the corresponding space of generalized functions. The bracket is then interpreted by the usual pairing of test functions and generalized functions.

In our interpretation no such distinction is made between bra and ket space. The roles of bras and kets can be interchanged, what we consider the main principle of the bracket formalism. So both bra and ket space are represented by spaces of generalized functions of the same type. However, our bracket is no longer a complex number but an analytic function in the open right half plane. This analytic function represents an (almost) periodic distribution. Thus we get a mathematical justification for the various heuristic formulae of Dirac. We mention $<\delta_x,\delta_y>=\delta_y(x)$.

Our contribution consists of three parts.

In this first part we introduce the mathematical concepts which lie at the basis of Dirac's formalism. They are described in the setting of Sobolev triples of Hilbert spaces. In this respect we mention the concept of Dirac basis, which we consider the natural measure theoretical generalization of the concept of orthonormal basis. The second part is devoted to our mathematical interpretation of Dirac's formalism. Having introduced the bra and ket space, we give an interpretation of the pairing between kets and bras, of expansions with respect to continuum sets of kets, of orthogonality of complete sets of eigenkets and of matrices with respect to these complete sets. A mathematical interpretation of the free field formalism can be found in the last part.

With this series of lectures we want to give a survey of the results we gathered the passed few years. Therefore, most proofs are omitted. The main reference is our monograph [EG1].

I. Dirac Bases in Sobolev Triples of Hilbert Spaces

1. Federer measure spaces

The measure spaces that we call Federer measure spaces have the differentiation property studied in Section 2.9 of Federer's monograph [Fe].

1.1. Definition

A σ-finite measure space (M,μ) is said to be a Federer measure space if it possesses the following properties.

- M is a separable topological space with a metric d;
- μ is a regular Borel measure on M such that bounded Borel subsets of M have finite μ-measure;
- The measure space (M,μ) admits the following differentiation theorem: Let ϕ: $M \to \mathbb{C}$ denote a Borel function which is integrable on bounded Borel sets. Then there exists a null set N_ϕ such that for all $r>0$ and all $x \in M \setminus N_\phi$ the *closed* ball $B(x,r)$ with radius r and centre x has positive μ-measure and, moreover, the limit

  $$\tilde{\phi}(x) = \lim_{r \downarrow 0} \mu(B(x,r))^{-1} \int_{B(x,r)} \phi d\mu$$

  exists for all $x \in M \setminus N_\phi$. Here the function $x \mapsto \tilde{\phi}(x)$ fixes a Borel function $\tilde{\phi}$ with $\phi = \tilde{\phi}$ μ-almost everywhere. We observe that for $x \in M \setminus N_\phi$ we have

  $$\tilde{\phi}(x) = \lim_{r \downarrow 0} \mu B(x,r))^{-1} \int_{B(x,r)} \tilde{\phi} d\mu. \qquad (*)$$

In literature there are given conditions on the metric d of a separable metric space M such that the measure space (M,μ) is a Federer measure space for *any* regular σ-finite Borel measure μ on M with the property that bounded Borel sets of M have finite μ-measure. These conditions are of a geometric nature. E.g. $(\mathbf{R}^n,\mu)$ is a Federer measure space for each σ-finite Borel measure μ which is bounded on bounded Borel sets. See [WZ] and [EG1], Section A.II.3.

The next theorem states the remarkable result that in the classical Sobolev lemma the 'open set in $\mathbf{R}^n$' can be replaced by 'Federer measure space', and the operator

$$\Delta=\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}$$

by *any* positive operator Δ in $L_2(M,\mu)$, which has a Hilbert-Schmidt inverse (or, more generally, by a positive operator Δ with a Carleman inverse, see [EG1], Section A.II.4.)

Let X denote a separable Hilbert space and $\mathcal{R}$ a positive Hilbert-Schmidt operator from X into X. We make $\mathcal{R}(X)$ into a Hilbert space by means of the inner product $(u,v)_1=(\mathcal{R}^{-1}u,\mathcal{R}^{-1}v)_X$ where $(\cdot,\cdot)_X$ denotes the inner product in X. Also we introduce the notation $\mathcal{R}^{-1}(X)$ For the completion of X with respect to the inner product $(f,g)_{-1}=(\mathcal{R}f,\mathcal{R}g)_X$. The pairing between $\mathcal{R}(X)$ and $\mathcal{R}^{-1}(X)$ is denoted by $<u,G>=(\mathcal{R}^{-1}u,\mathcal{R}G)_X$.

Let (M,μ) denote a Federer measure space and let P: $X\to L_2(M,\mu)$ denote a densely defined linear operator with $\mathcal{R}(X)$ contained in its domain $D(\mathcal{P})$. All this is gathered in the following diagram

$$\begin{array}{ccccc}\mathcal{R}(X) & \hookrightarrow & X & \hookrightarrow & \mathcal{R}^{-1}(X)\\ & & \downarrow\mathcal{P} & & \\ & & L_2(M,\mu) & & \end{array}$$

1.2. Theorem (Measure theoretical Sobolev lemma).
Let $\mathcal{P}$ and $\mathcal{R}$ be such that $\mathcal{P}\mathcal{R}$: $X\to L_2(M,\mu)$ is a Hilbert-Schmidt operator. Let $(u_k)_{k\in\mathbf{N}}$ denote an orthonormal basis of eigenvectors of $\mathcal{R}$ with eigenvalues $\rho_k>0$. For each $w\in\mathcal{R}(X)$ there exists a representant $(\mathcal{P}w)^\sim$ of $\mathcal{P}w\in L_2(M,\mu)$ such that

(a) *There exists a null set $N\subseteq M$ such that for all $x\in M\setminus N$, $w\in\mathcal{R}(X)$*

$$(\mathcal{P}w)^\sim(x) = \lim_{r\downarrow 0}\mu(B(x,r))^{-1}\int_{B(x,r)}(\mathcal{P}w)^\sim d\mu$$

(b) *For each $x\in M$ the linear functional $w\mapsto(\mathcal{P}w)^\sim(x)$ is continuous on $\mathcal{R}(X)$; its Riesz representative in $\mathcal{R}(X)$ equals*

$$e_x = \sum_{k=1}^{\infty}\rho_k^2\overline{(\mathcal{P}u_k)^\sim(x)}u_k$$

So $(\mathcal{P}w)^\sim(x) = (W,e_x)_1$.

(c) *Suppose in addition that the function $\Sigma_{k=1}^{\infty}|(\mathcal{P}\mathcal{R}u_k)^\sim|^2$ is essentially bounded on M. Then there exists a null set N_0 such that*

$$\exists_{L>0}\forall_{w\in\mathcal{R}(X)}\forall_{x\in M\setminus N_0}:\ |(\mathcal{P}w)^\sim(x)|\leq L\|w\|_1$$

(d) *Suppose in addition that the function k: $M\to X$,*

$$k(x) = \sum_{k=1}^{\infty} \rho_k (\mathcal{P}u_k)^{\sim}(x) u_k$$

is continuous at $x = a$*. Then for each* $w \in \mathcal{R}(X)$ *the function* $(\mathcal{P}w)^{\sim}$ *is continuous at* $x = a$.

Remarks

- For the set $M \setminus N$ one can take the set of all $x \in M$ for which the relative differentiation result(*) holds with $\tilde{\phi}$ replaced by $(u_k)^{\sim}|$, $|(\mathcal{P}u_k)^{\sim}|$ and $\Sigma_{k=1}^{\infty} \rho_k^2 |(\mathcal{P}u_k)^{\sim}|^2$
- The proof of the previous theorem can be found in [EG2]. The condition that $\mathcal{R}$ is a positive Hilbert-Schmidt operator and $\mathcal{P}\mathcal{R}$ a Hilbert-Schmidt operator can be relaxed by taking $\mathcal{R}$ any positive bounded operator and $\mathcal{P}\mathcal{R}$ a bounded Carleman operator from X into $L_2(M, \mu)$.

2.3. Application (δ-functions on Federer measure spaces).
Let (M, μ) be a Federer measure space and let $\mathcal{R}$ be a positive Hilbert-Schmidt operator on $L_2(M, \mu)$. Then by Theorem 1.2 there are continuous linear functionals l_x, $x \in M$, and a null set N such that for each $\phi \in \mathcal{R}(L_2(M, \mu))$,

$$\tilde{\phi}: x \mapsto l_x(\phi) \quad \text{is a representative of} \quad \phi$$

and

$$\tilde{\phi}(x) = \lim_{r \downarrow 0} \mu(B(x,r))^{-1} \int_{B(x,r)} \phi d\mu, \quad x \in M \setminus N.$$

$\tilde{\phi}$ may be called a canonical representative.

Since the Hilbert spaces $\mathcal{R}(L_2(M, \mu))$ and $\mathcal{R}^{-1}(L_2(M, \mu))$ are in duality, there exist $\delta_x \in \mathcal{R}^{-1}(L_2(M, \mu))$, $x \in M$, such that $l_x(\phi) = <\phi, \delta_x>$. We say that $\mathcal{R}^{-1}(L_2(M, \mu))$ contains a complete set of delta functions.

2.4. Application (The classical Sobolev embedding theorem on $[-\pi, \pi]^n$).
In $L_2([-\pi, \pi]^n, dx)$ we consider the operator Δ,

$$\Delta = (1 - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2})$$

where we impose periodic boundary conditions. For $m \in \mathbb{N}$ we put $\mathcal{R}_m = \Delta^{-m/2}$ and $\mathcal{H}_m = \mathcal{R}_m(L_2([-\pi, \pi]^n, dx))$. Then from Theorem 1.2 we derive.

Let $m > n/2$ and let $0 \leq l < m - n/2$, $l \in \mathbb{N} \cup \{0\}$. Then there is a null set $N_m^{(l)}$ such that for each $u \in \mathcal{H}_m$ there exists a representative $(u)^{\sim}$ with the property that for all $\sigma \in (\mathbb{N} \cup \{0\})^n$, $\sigma_1 + \cdots + \sigma_n \leq l$, there exists $\gamma_\sigma > 0$ independent of u such that

$$\forall_{x \in [-\pi, \pi]^n \setminus N_m^{(l)}}: |(D^\sigma (u)^{\sim})(x)| \leq \gamma_\sigma \|u\|_m$$

Here D^σ denotes the differential operator $(\frac{\partial}{\partial x_1})^{\sigma_1} \cdots (\frac{\partial}{\partial x_n})^{\sigma_n}$ and $\|\cdot\|_m$ the norm of $\mathcal{H}_m$. That is $\|u\|_m = \Delta^{m/2} u\|_{L_2}$.

2. Dirac bases

Let X be a separable Hilbert space and let $(v_n)_{n\in\mathbb{N}}$ be an orthonormal basis in X. The choice of $(v_n)_{n\in\mathbb{N}}$ fixes a unitary operator Φ: $X\to l_2$, $\Phi f=(n\mapsto(f,v_n))$ and vice versa.

From a measure theoretical point of view the Hilbert space l_2 consists of square integrable functions from $\mathbb{N}$ into $\mathbb{C}$ with respect to the counting measure τ defined by $\tau(A)=\#A$, $A\subseteq\mathbb{N}$. An orthonormal basis $(v_n)_{n\in\mathbb{N}}$ in X is an X-valued function from $\mathbb{N}$ into X such that $n\mapsto(f,v_n)$ is a Borel function on $\mathbb{N}$ for each $f\in X$ and

$$\forall_{f\in X}\forall_{g\in X}\colon \int_{\mathbb{N}}(f,v_n)_X(g,v_n)_X d\tau(n) = (f,g)_X.$$

In our concept of Dirac basis the measure space $(\mathbb{N},\tau)$ is replaced by a general measure space (M,μ). Thus the concept of Dirac basis is a 'continuum' substitute of the discrete concept of orthonormal basis.

Let $\mathfrak{R}$ denote a positive Hilbert-Schmidt operator on X and (M,μ) an arbitrary σ-finite measure space. We recall that a function Θ: $M\to\mathfrak{R}^{-1}(X^-)$ is called a Borel function if for each $w\in\mathfrak{R}(X)$ the function $x\mapsto\langle w,\Theta(x)\rangle$ is a Borel function on N. An $\mathfrak{R}^{-1}(X)$-valued Borel function Θ on M is called weakly integrable if for all $w\in\mathfrak{R}(X)$ the function $x\mapsto\langle w,\Theta(x)\rangle$ is μ-integrable; Θ is called strongly integrable if in addition the function $x\mapsto\|\Theta(x)\|_{-1}$ is integrable. In the latter case, the linear functional

$$w\mapsto\int_M\langle w,\Theta(x)\rangle d\mu(x),\quad w\in\mathfrak{R}(X)$$

is continuous. We denote its Riesz representative in $\mathfrak{R}^{-1}(X)$ by $\int_M\Theta(x)d\mu(x)$. We note that

$$\|\int_M\Theta(x)d\mu(x)\|_{-1} \leq \int_M\|\Theta(x)\|_{-1}d\mu(x).$$

2.1. Definition

Let (M,μ) denote a σ-finite measure space, X a separable Hilbert space and $\mathfrak{R}$ a positive Hilbert-Schmidt operator on X. Let $(u_k)_{k\in\mathbb{N}}$ be an orthonormal basis in X, which consists of eigenvectors of $\mathfrak{R}$ with corresponding eigenvalues $\rho_k>0$. Finally, let $[G]$ denote an equivalence class of Borel functions F: $M\to\mathfrak{R}^{-1}(X)$

(a) $([G], M, \mu, \mathfrak{R}, X)$ is called a *Dirac basis* if for $\hat{G}\in[G]$ the following relations are valid

$$\int_M\langle u_l,\hat{G}(x)\rangle\,\langle u_k,\hat{G}(x)\rangle d\mu(x) = \delta_{kl},\quad k,l\in\mathbb{N}.$$

(It follows that the function

$$x\mapsto\sum_{k=1}^{\infty}\rho_k^2|\langle u_k,\hat{G}(x)\rangle|^2 = \|G(x)\|_{-1}^2$$

is integrable and hence for all $w \in \mathfrak{R}(X)$ the function $x \mapsto <w, \hat{G}(x)>$.)

(b) In addition, assume (M, μ) is a Federer measure space. A representative $\tilde{G} \in [G]$ is called a canonical Dirac basis if there exists a null set $\tilde{N} \subseteq M$ such that for all $w \in \mathfrak{R}(X)$ and all $x \in M \setminus \tilde{N}$

$$\lim_{r \downarrow 0} \mu(B(x,r))^{-1} \int_{B(x,r)} <w, \tilde{G}(y)> d\mu(y) = <w, \tilde{G}(x)>$$

For a canonical Dirac basis we use the notation $(\tilde{G}_x)_{x \in M}$.

Example. Each orthonormal basis in X is a canonical Dirac basis.

Let $([G], M, \mu, \mathfrak{R}, X)$ be a Dirac basis with representative $\hat{G} \in [G]$. The functions $x \mapsto \phi_k(x) = <u_k, \hat{G}(x)>$ establish an orthonormal system $([\phi_k])_{k \in \mathbb{N}}$ in $L_2(M, \mu)$. So the linear operator $V: X \to L_2(M, \mu)$ defined by

$$Vf = \sum_{k=1}^{\infty} (f, u_k)[\phi_k]$$

is an isometry. From this observation follows the Plancherel formula for all $w, v \in \mathfrak{R}(X)$,

$$(w, v)_X = \int_M <w, \hat{G}(x)> <v, \hat{G}(x)> d\mu(x).$$

Also the converse

2.2. Theorem

Let $V: X \to L_2(M, \mu)$ be an isometry.

(a) *There exists a Dirac basis $([G], M, \mu, \mathfrak{R}, X)$ such that for each $\hat{G} \in [G]$ and all $w \in \mathfrak{R}(X)$ the function $x \mapsto <w, \hat{G}(x)>$ is a representative of Vw.*

(b) *In addition, assume (M, μ) is a Federer measure space. Then there exists a canonical Dirac basis $(\tilde{G}_x)_{x \in M}$ and a null set $\tilde{N}$ such that*

- $\forall_{w \in \mathfrak{R}(X)}: (x \mapsto <w, \tilde{G}_x>) \in V_w$
- $\lim_{r \downarrow 0} \| \tilde{G}_x - \mu(B(x,r))^{-1} V^* \chi_{B(x,r)} \|_{-1} = 0$

Here $\chi_{b(x,r)}$ denotes the characteristic function of the closed ball $B(x,r)$.

It would not be proper to call a new mathematical notion a basis if there were no expansion result. For Dirac bases we have the following type of expansions.

2.3. Theorem

Let $([G], M, \mu, \mathfrak{R}, X)$ be a Dirac basis. Then for each $w \in \mathfrak{R}(X)$, the $\mathfrak{R}^{-1}(X)$-valued function

$$x \mapsto <w, \hat{G}(x)> \hat{G}(x)$$

is strongly integrable and

$$w = \int_M <w, \hat{G}(x)> \hat{G}(x) d\mu(x).$$

There is a strong connection between the notion of canonical Dirac basis and the generalized eigenvalue problem: any canonical Dirac basis, which is related to a unitary operator $U\colon X \mapsto L_2(M,\mu)$ as indicated in Theorem 2.2, consists of generalized eigenvectors of operators out of a commutative unbounded *-algebra of normal operators.

2.4. Theorem

(a) *Let (M,μ) be a Federer measure space, V a linear isometry from X into $L_2(M,\mu)$ and $h\colon M \to \mathbb{C}$ a Borel function which is bounded on bounded Borel sets. Then there exists a canonical Dirac basis $(\tilde{G}_x)_{x\in M}$ and a null set $N_h \subseteq M$ Such that for all $x \in M \setminus N_h$*

$$\lim_{r\downarrow 0} \|h(x)\tilde{G}_x - \mu B(x,r))^{-1}(V^* \mathfrak{M}_h \chi_{B(x,r)}\|_{-1} = c$$

Here $\mathfrak{M}_h$ denotes the multiplication operator in $L_2(M,\mu)$: $(\mathfrak{M}_h g)(x) = h(x)g(x)$ with its maximal domain.

(b) *In addition, suppose that $V\colon X \to L_2(M,\mu)$ is a unitary operator and that $V^*\mathfrak{M}_h V$ is closable in $\mathfrak{R}^{-1}(X)$, i.e. $\mathfrak{R}V^*\mathfrak{M}_h V\mathfrak{R}^{-1}$ is closable in X. Let $\overline{V^*\mathfrak{M}_h V}$ denote the $\mathfrak{R}^{-1}(X)$-closure of $V^*\mathfrak{M}_h V$. Then the canonical Dirac basis of (a) satisfies $(\overline{V^*\mathfrak{M}_h V})\tilde{G}_x = h(x)\tilde{G}_x$.*

2.5. Example

Consider $M=[0,\pi]$ with μ the usual Lebesque measure. So we take $X=L_2([0,\pi])$, For $\mathfrak{R}$ we choose the positive Hilbert-Schmidt operator $(-\frac{d^2}{dx^2})^{-1/2}$ (zero boundary conditions). Then $\mathfrak{R}u_k = \frac{1}{k}u_k$, $k\in\mathbb{N}$, with $u_k(x) = \frac{\sqrt{2}}{\pi}\sin kx$.

Let V denote the identity mapping in $L_2([0,\pi])$. In this case

$$\tilde{G}_y = \frac{2}{\pi}\sum_{k=1}^{\infty} \sin(ky)u_k, \quad y\in[0,\pi].$$

This is the ordinary expression $\delta_y(x) = \frac{2}{\pi}\Sigma_{k=1}^{\infty}\sin ky \sin kx$. We observe that $\delta(y-x)$ is meaningless in this case! Next we consider $h(x)=x$. Since the operators $\mathfrak{R}^{-1}\cdot\mathfrak{M}_h\cdot\mathfrak{R}$ and $\mathfrak{R}\cdot\mathfrak{M}_h\cdot\mathfrak{R}^{-1}$ are densely defined (the linear span $<\{u_k | k\in\mathbb{N}\}>$ is contained in their domains) the operator $\mathfrak{M}_h$ is closable in $\mathfrak{R}^{-1}(X)$ and hence $\tilde{G}_y$ is a genuine generalized eigenfunction of $\mathfrak{M}_h$.

2.6. Example

We take $X=L_2(\mathbb{R})$, $\mathfrak{R}=(\frac{1}{2}(-\frac{d^2}{dx^2}+x^2+1))^{-1}$, $V=\mathbb{F}$, the Fourier transformation on $L_2(\mathbb{R})$ and $h\colon x\mapsto x$. The Hermite functions $(\psi_k)_{k\in\mathbb{N}\cup\{0\}}$ establish an orthonormal basis in X with $\mathfrak{R}\psi_k = (k+1)^{-1}\psi_k$, $k\in\mathbb{N}\cup\{0\}$. So in this case

$$\tilde{G}_y = \sum_{k=0}^{\infty} \overline{(\mathbb{F}\psi_k)(y)}\psi_k.$$

This series converges relatively. We have

$$\tilde{G}_y(x) = \sum_{k=0}^{\infty} \overline{(\mathbb{F}\psi_k)(y)}\psi_k(x) = \sum_{k=0}^{\infty} (i)^k \psi_k(y)\psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ixy}$$

These are indeed eigenfunctions of $\mathbb{F}^* \mathfrak{M}_h \mathbb{F} = -i\frac{d}{dx}$. The closedness condition can be verified using the fact that the operator $\mathfrak{M}_h$ has a column finite (and hence row finite) matrix representation with respect to the basis $(\psi_k)_{k \in \mathbb{N} \cup \{0\}}$.

3. The generalized eigenvalue problem for self-adjoint operators

In Theorem 2.4 for the Federer measure space (M,μ) we take the disjoint union of a countable number of copies of $\mathbb{R}^n$ with a finite nonnegative Borel measure on each copy. So for a countable set $\mathbb{D}$ we have $M = \mathbb{R}^n \times \mathbb{D}$. A Borel subset of M is of the form $\cup_{d \in \mathbb{D}} B_d \times \{d\}$ with B_d, $d \in \mathbb{D}$, Borel subsets of $\mathbb{R}^n$. Given finite nonnegative Borel measures $\mu^{(d)}$, we write $\mu = \oplus_{d \in \mathbb{D}} \mu^{(d)}$ indicating the Borel measure on M defined by

$$\mu(\bigcup_{d \in \mathbb{D}} B_d \times \{d\}) = \sum_{d \in \mathbb{D}} \mu^{(d)}(B_d).$$

So we have $L_2(M,\mu) = \oplus_{d \in \mathbb{D}} L_2(\mathbb{R}^n, \mu^{(d)})$. We define the function h_j, $j = i, \ldots, n$, on M by

$$h_j(x,d) = x_j, \quad x \in \mathbb{R}^n, \quad d \in \mathbb{D}.$$

Then the multiplication operator $Q_j = \mathfrak{M}_{h_j}$ is the self-adjoint operator of multiplication by the j-th coordinate function $x \mapsto x_j$ in each direct summand $L_2(\mathbb{R}^n, \mu^{(d)})$. Now, if $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$ is an n-set of mutually commuting self-adjoint operators in a separable Hilbert space X, it follows from spectral theory that there exists an almost countable number of finite nonnegative Borel measures $\mu^{(d)}$ and a unitary operator V: $X \to \oplus_{d \in \mathbb{D}} L_2(\mathbb{R}^n, \mu^{(d)})$ such that $\mathcal{P}_j = V^* Q_j V$, $j = 1, \ldots, n$. Then we can apply Theorem 2.4.

It is possible however to bring in much more delicacy.

3.1. Definition

The n-set $(T_1, \cdots, T_n)$ of commuting self-adjoint operators in a separable Hilbert space Y is called of uniform multiplicity m, $1 \leq m \leq \infty$, if there exists a finite nonnegative Borel measure μ on $\mathbb{R}^n$ and a unitary operator U.

$$U: Y \to \bigoplus_{l=1}^{m} L_2(\mathbb{R}^n, \mu),$$

such that UT_jU^* equals multiplication by the j-th coordinate function in each direct summand.

Remark

If $m = 1$, then, in Quantum Mechanics, the n-set $(T_1, \ldots, T_n)$ is called a complete set of observables.

In a finite dimensional Hilbert space E each commuting n-set of self-adjoint operators $(\mathfrak{B}_1, \ldots, \mathfrak{B}_n)$ has a complete set of simultaneous eigenvectors. An element $\lambda \in \mathbf{R}^n$ is called an eigentuple of the n-set $(\mathfrak{B}_1, \ldots, \mathfrak{B}_n)$ if there exists a vector $e_\lambda \in E$ such that

$$\mathfrak{B}_j e^\lambda = \lambda_j e_\lambda, \quad j = 1, \ldots, n.$$

The set of all eigentuples of $(\mathfrak{B}_1, \ldots, \mathfrak{B}_n)$ may be called the joint spectrum of $(\mathfrak{B}_1, \ldots, \mathfrak{B}_n)$ denoted by $\sigma(\mathfrak{B}_1, \ldots, \mathfrak{B}_n)$. In order to list all eigentuples in a well-ordered manner one can list all eigentuples of multiplicity one, two, etc. In fact, this is precisely the outcome of the following theorem for the infinite dimensional case.

3.2. Theorem (commutative multiplicity theorem)
Let $(\mathfrak{P}_1, \ldots, \mathfrak{P}_n)$ denote an n-set of commuting self-adjoint operators in a separable Hilbert space X. Then X can be split into a direct sum,

$$X = X_\infty \oplus X_1 \oplus X_2 \oplus \cdots,$$

the so-called standard splitting, such that the following assertions are valid.
(a) *The n-set $(\mathfrak{P}_1, \ldots, \mathfrak{P}_n)$ restricted to X_m, $m = \infty, 1, 2, \ldots$, acts invariantly in X_m and has uniform multiplicity m.*
(b) *The finite nonnegative Borel measures μ_m corresponding to each X_m are mutually disjoint, i.e. $\mu_k \perp \mu_l$ if $k \neq l$ which means $\mu_k(supp(\mu_k) \cap supp(\mu_l)) = \mu_l(supp(\mu_k) \cap supp(\mu_l)) = 0$.*
(Here $supp(\mu)$ denotes the complement of the largest open set of μ-measure zero).

We want to apply Theorem 2.4.6. Therefore we need

3.3. Lemma
Let $(\mathfrak{P}_1, \ldots, \mathfrak{P}_n)$ be an n-set of mutually commuting self-adjoint operators in the separable Hilbert space X. Then there exists a positive Hilbert-Schmidt operator $\mathfrak{R}$ on X such that each operator $\mathfrak{R}\mathfrak{P}_l\mathfrak{R}^{-1}$, $l = 1, \ldots, n$, is densely defined and closable in X.

In the proof of the preceding lemma we construct an orthonormal basis $(u_k)_{k \in \mathbf{N}}$ in X such that each matrix $((\mathfrak{P}_j u_k, u_l)_X)_{k \in \mathbf{N}}$ is column finite. Then we take $\mathfrak{R} = \Sigma_{k=1}^{\infty} \rho_k u_k \otimes u_k$, where $(\rho_k)_{k \in \mathbf{N}}$ can be any positive l_2-sequence.

We are now in a position to formulate our main theorem on the solution of the generalized eigenvalue problem for a finite number of commuting operators.

3.4. Theorem
Let $(\mathfrak{P}_1, \ldots, \mathfrak{P}_n)$ denote an n-set of commuting self-adjoint operators in a separable Hilbert space X.
(a) *There exists a positive Hilbert-Schmidt operator $\mathfrak{R}$ such that the operators $\mathfrak{P}_j$ extend to closed operators $\overline{\mathfrak{P}}_j$ in the Hilbert space $\mathfrak{R}^{-1}(X)$.*
(b) *Let $X = X_\infty \oplus X_1 \oplus X_2 \oplus \cdots$ be the standard splitting of X and $\mu_\infty, \mu_1, \mu_2, \ldots$ the corresponding multiplicity measures. Let $m = \infty, 1, 2, \ldots$.*

Then there is a μ_m-null set N_m with the following property: for all $x=(x_1,\ldots,x_n)\in supp(\mu_m)\setminus N_m$ there exist m independent vectors $\tilde{E}^{(m)}_{x,l}\in \mathfrak{R}^{-1}(X)$, $1\leq l<m+1$, satisfying $\overline{\mathcal{P}_j}\tilde{E}^{(m)}_{x,l}=x_j\tilde{E}^{(m)}_{x,l}$, $j=1,\ldots,n$.

(c) $\sigma(\mathcal{P}_1,\ldots,\mathcal{P}_n)=\cup_{m=1}^{\infty} supp(\mu_m)\cup supp(\mu_\infty)$

(d) *The set $\{\tilde{E}^{(m)}_{x,l}\,|\,m=\infty,1,2,\ldots,\ 1\leq l\leq m,\ x\in supp(\mu_m)\}$ establishes a canonical Dirac basis in $\mathfrak{R}^{-1}(X)$.* □

Remarks

- The proofs of Lemma 3.3 and Theorem 3.4 can be found in [EG3]. In [EG1] we give a generalized version of Theorem 3.4 based on Carleman operators.
- Since for the eigenvalues of $\mathfrak{R}$ *any* positive l_2-sequence can be taken, it clear that the improper eigenvectors of the operators $\mathcal{P}_1,\ldots,\mathcal{P}_n$ lie at the 'periphery' of the Hilbert space X.

3.5. Example

Consider $X=L_2(\mathbb{R})$ and $\mathfrak{R}=(x^2-\frac{d^2}{dx^2})^{-1}$. Let Q denote the operator of multiplication by the identity function with its maximal domain, and π the parity operator. The pair of operator (Q^2,π) establishes a complete set of commuting operators. Furthermore, the operators Q^2 and π are closable in $\mathfrak{R}^{-1}(X)$ with closures denoted by $\overline{Q^2}$ and $\overline{\pi}$.

Let $(\tilde{G}_y)_{y\in\mathbb{R}}$ be the canonical Dirac basis introduced in Example 2.6. Then we have

$$\begin{aligned}\overline{Q^2}\tilde{G}_y &= y^2\tilde{G}_y\\ \overline{\pi}\tilde{G}_y &= \mathrm{sign}(y)\tilde{G}_y,\quad y\in\mathbb{R}\setminus\{0\}.\end{aligned}$$

3.6. Example

Let $\mathcal{P}$ be a self-adjoint operator in X and $\mathfrak{R}$ a positive Hilbert-Schmidt operator such that $\mathfrak{R}\mathcal{P}\mathfrak{R}^{-1}$ is closable in X. The spectrum of $\mathfrak{R}\mathcal{P}\mathfrak{R}^{-1}$ can be larger than the spectrum of $\mathcal{P}$. An interesting example is the following. In $L_2(\mathbb{R})$ take $\mathcal{P}=i\frac{d}{dx}$, then $\sigma(\mathcal{P})=\mathbb{R}$. Further take $\mathfrak{R}=\exp(-\tau\mathcal{H})$ where $\mathcal{H}=x^2-\frac{d^2}{dx^2}$ and $\tau>0$. Then $\mathfrak{R}\mathcal{P}\mathfrak{R}^{-1}=i\cosh\tau\frac{d}{dx}+ix\sinh\tau$. Each $\lambda\in\mathbb{C}$ is an eigenvalue of this operator. Its eigenvector is

$$x \mapsto \exp((-i\lambda/\cosh\tau)x-\tfrac{1}{2}(\tanh\tau)x^2)$$

which belongs to $L_2(\mathbb{R})$. This set of eigenvectors is closely related to the set of so-called coherent states.

II. A Mathematical Interpretation of Dirac's Bracket Formalism

In his famous bracket formalism, Dirac considers two kinds of vectors, the kets and the bras. He assumes a one-one correspondence between ket and bra vectors. Also a number of algebraic relations are supposed to be satisfied. However, the term algebraic seems to be misused. Without excuse the 'algebraic' relations often involve infinite sums and integrals. So in a mathematical interpretation one is forced to take a *topological* vector space for the ket space K. In addition we want to remain as close to Hilbert space as possible. For this we have the following reasons.

- Dirac supposes that there are kets to which a 'finite' 'length' can be attached. These normalizable kets establish an infinite dimensional subspace N of K
- The scalar product of a normalizable bra and a normalizable ket is a complex number. So it is natural to assume that the norm of N arises from an inner product
- Dirac supposes that the nonnormalizable ket can, one way or another, be approximated by normalizable kets. Rephrased in mathematical terms: N must be dense in K
- Dirac's observables are real dynamical variables with real eigenvalues and a complete set of eigenstates. This leads to the mathematical concept of self-adjoint operator in a Hilbert space.

Throughout we use Dirac's bracket notation.

1. Kets

For the ket space we take a nuclear trajectory space $T_{X,A}$. Such a space is fixed by a pair (X,A): X is a separable Hilbert space and A a positive unbounded self-adjoint operator in X such that for all $t>0$ the operator e^{-tA} is Hilbert-Schmidt. This implies that there exists an orthonormal basis of eigenvectors v_d,

$d \in \mathbb{D}$, $\mathbb{D}$ a countable index set, of A with corresponding eigenvalues λ_d satisfying $\forall_{t>0}$: $\Sigma_{d\in\mathbb{D}} e^{-\lambda_d t} < \infty$.

1.1. Definition

Let the pair (X,A) be fixed as indicated above. An X-valued function $F: (0,\infty) \to X$ with the property

$$\forall_{t>0} \forall_{\tau>0}: F(t+\tau) = e^{-\tau A} F(t)$$

is called a trajectory. The set of all these trajectories establishes a complex vector space denoted by $T_{X,A}$. In the sequel the elements of $T_{X,A}$ will be called kets and will be denoted by $|K>$. Here K may indicate any label(s).

1.2. Notation

Let $t'>0$. By $e^{-t'A}|K>$ we mean the ket $t \mapsto |K>(t+t')$, $t>0$.

1.3. Definition

If $\lim_{t\downarrow 0}|K>(t)=g$ exists in X-sense we say that $|K>$ is a normalizable ket. We write $|K>(0)=g$. Then $|K>(t)=e^{-tA}g$. Further, we define $\| \, |K> \| = \|g\|_X$.

Not every ket is normalizable. Consider e.g.

$$t \mapsto Ae^{-tA}f, \quad f\in X, \quad f\notin D(A).$$

1.4. Definition

A ket $|W>$ is called a test ket if there exists $\sigma>0$ and a ket $|K>$ such that $|W>=e^{-\sigma A}|K>$. It follows that a test $|W>$ can be extended to the interval $(-\sigma,\infty)$ with $|W>(t)=|K>(t+\sigma)$, $t>-\sigma$. The set of test kets establishes a linear subspace of $T_{X,A}$ and is denoted by $S_{X,A}$.

We observe that a ket $|W>$ is a test ket iff $|W>$ is normalizable with $|W>(0)\in D(e^{\tau A})$ for certain $\tau>0$. In this case $|W>$ extends to an X-valued analytic function on a neighbourhood of 0 and $|W>(0)$ is an analytic vector for the operator A. For this reason $S_{X,A}$ is called an analyticity space. The operators e^{-tA}, $\mathrm{Re}\, t>0$, constitute a holomorphic semigroup. Therefore, any ket $|K>$ extends to an X-valued holomorphic function on the open right half plane $\mathrm{Re}\, t>0$ of the complex t-plane. If $|K>$ is normalizable it, in addition, extends to a continuous function on the *closed* right half plane $\mathrm{Re}\, t\geqslant 0$. If $|K>$is a test ket it extends analytically to the open half plane $\mathrm{Re}\, t>-\sigma$ for some $\sigma>0$ dependent on $|K>$.

The orthonormal basis $(V_d)_{d\in\mathbb{D}}$ of eigenvectors of A consists obviously of analytic vectors. We denote the corresponding test kets by $|v,d>$. So $|v,d>(t)=e^{-tA}v_d=e^{-t\lambda_d}v_d$ and for t any real (complex) number can be taken.

For any ket $|K>$ consider the expansion

$$|K>(t) = \sum_{d\in\mathbb{D}} (|K>(t),v_d)_X v_d) = \sum_{d\in\mathbb{D}} (|K>(t),|v,d>(-t))_X |v,d>(t).$$

The expression

$$\beta_d = (|K>(t),|\nu,d>(-t))_X$$

does not depend on t, whence the β_d are well defined complex numbers. They are the expansion coefficients for kets in the sense that

$$|K> = \sum_{d\in\mathbf{D}} \beta_d|\nu,d>,$$

which means

$$\forall_{t>0}: |K>(t) = \sum_{d\in\mathbf{D}} \beta_d|\nu,d>(t)$$

where the latter series converges in X.

1.5. Theorem

A complex sequence $(\beta_d)_{d\in\mathbf{D}}$ corresponds to a ket $|K>=\Sigma_{d\in\mathbf{D}}\beta_d|\nu,d>$ iff $\forall_{t>0}$: $\Sigma_{d\in\mathbf{D}}|\beta_d|^2e^{-t\lambda_d}<\infty$. The ket $|K>$ is normalizable iff $\Sigma_{d\in\mathbf{D}}|\beta_d|^2<\infty$, and a test ket iff $\exists_{\tau>0}$: $\Sigma_{d\in\mathbf{D}}|\beta_d|^2e^{-t\lambda_d}<\infty$.

2. BRAS

Let X' denote the topological dual of X. The Riesz representation theorem says that there exists an anti-linear one-one correspondence between X and X', which we denote by $'$. So $X\ni f\leftrightarrow f'\in X'.X'$ is also a Hilbert space and

$$(g',f')_{X'} = (f,g)_X = g'(f) = \overline{f'(g)}.$$

Let L be a linear operator in X. We define the corresponding operator L' in X' by

$$(L'g')(f) = g'(Lf), \quad g'\in X', \quad f\in X.$$

So $\forall_{f\in X}$: $(L'g')(f)=g'(Lf)=(Lf,g)_X$. For each $g'\in X'$ the functional $L'g'$ corresponds to a vector g_L in X. Thus we obtain the linear operator L^*: $g\mapsto g_L$, $g\in X$. We have, replacing L by L^*, for all f, $g\in X$

$$((L^*)'g')(f) = g'(L^*f)=(L^*f,g)_X=(g,Lg)_X=(Lg)'(f).$$

It follows that $(Lg)'=(L^*)'g'$. In all these expressions the usual care with the domains must be taken. Similar to the ket case we introduce the triple of spaces

$$S_{X',A'} \hookrightarrow X' \hookrightarrow T_{X',A'}.$$

The elements of $T_{X',A'}$ are called bras. We denote them by $<B|$. So any bra $<B|$ is a function from $(0,\infty)$ into X' with the property

$$e^{-t'A'}<B|(t) = <B|(t+t'), \quad t,t'>0.$$

By $<B|e^{-\tau A}$ we mean the bra $t\mapsto<B|(t+\tau)$. The elements of X' represent the normalizable bras; to $g'\in X'$ corresponds the bra $<B|$: $t\mapsto e^{-tA'}g'$. The elements of $S_{X',A'}$ are the test bras.

Since $(e^{-tA})^*=e^{-tA}$, for all $g\in X$ we have $(e^{-tA}g)'=e^{-tA'}g'$. So there is a one-one correspondence between kets and bras. To each ket $|K>$ corresponds the bra $<K|$ defined by

$$<K|: t \mapsto (|K>(t))'$$
$$((|K>(t+t'))' = (e^{-t'A}|K>(t))' = e^{-t'A}(|K>(t))'.$$

Thus, the space of normalizable bras is in one-one correspondence with the space of normalizable kets and the space of test bras is in one-one correspondence with the space of test kets.

If we define the test bra $<v,d|$ by $<v,d|(t) = e^{-tA'}\, v_d' = e^{-t\lambda_d} v_d'$, we have the expansion

$$<B| = \sum_{d \in \mathbf{D}} \beta_d <v,d|$$

with the same interpretation as given in Theorem 1.5.

3. BRACKETS

With any given bra $<B|$ and any given ket $|K>$ we associate the bracket $<B|K>$.

3.1. Definition

The bracket $<B|K>$ denotes the complex valued function on $(0,\infty)$ defined by

$$<B|K>(t) = (<B|(\tau)(|K>(t-\tau))$$

where for each $t>0$ any τ, $0<\tau<t$, can be taken.

3.2. Theorem

(a) *If* $<B = \Sigma_{d \in \mathbf{D}} \xi_d <v,d|$ *and* $|K> = \Sigma_{d \in \mathbf{D}} \zeta_d |v,d>$ *then* $<B|K>(t) = \Sigma_{d \in \mathbf{D}} \xi_d \zeta_d e^{-t\lambda_d}$

(b) *The bracket* $<B|K>$ *extends to an analytic function on the open right half plane* $Re\, t>0$

(c) *If in addition,* $<B|$ *and* $|K>$ *are normalizable, then* $<B|K>$ *extends to a continuous function on the closed right half plane*

(d) *If* $<B|$ *is a test bra or* $|K>$ *is a test ket, then* $<B|K>$ *extends to an analytic function on the halfplane* $\mathrm{Re}\, t > -\sigma$ *for some* $\sigma>0$ *dependent on* $<B|$ *or* $|K>$.

For any ket $|K>$ *let* $<K>$ *denote the bra corresponding to* $|K>$. *The following relations can be verified*

$$<K|K> \geqslant 0, \quad \text{i.e.} \quad \forall_{t>0}: <K|K>(t) \geqslant 0 \tag{3.3}$$
$$<K_1|K_2> = \overline{<K_1|K_2>}.$$
$$<B|\{|K_1>+|K_2>\} = <B|K_1> + <B|K_2>.$$

Remarks

- $<B|K>(t+iy)=\Sigma_{d\in\mathbf{D}}\xi_d\zeta_d e^{-t\lambda_d}e^{iy\lambda d}$ can be regarded as an almost periodic distribution on the imaginary axis
- Suppose $<B|: t\mapsto e^{-tA'}g',\ g'\in X'$ and $|K>: t\mapsto e^{-tA}f,\ f\in X$. Then $<B|K>(0)=(f,g)_X$.
- If, in addition, $f=Qg$, Q a observable, then the function

$$y\mapsto <B|K>(iy)$$

is the characteristic function of the probability distribution of a measurement of the observable Q if the quantum mechanical system is in the state g.

4. Linear operators

We denote the vector space of Hilbert-Schmidt operator from X into itself by $B_2(X)$. Note that $B_2(X)$ is again a Hilbert space.

4.1. Definition

$TT_{B_2(X),A\otimes I,I\otimes A}$ denotes the space which consists of all operator valued functions Θ: $(0,\infty)\times(0,\infty)\rightarrow B_2(X)$ which satisfy

$$\forall_{t,\tau>0}\forall_{s,\sigma>0}: \hat{\Theta}(t+\tau,s+\sigma) = e^{-\sigma A}\hat{\Theta}(t,s)e^{-\tau A}$$

The action of $TT_{B_2(X),A\otimes I,I\otimes A}$ on $S_{X,A}$ is defined by

$$\Theta|W>: s\mapsto\hat{\Theta}(\tau,s)(|W>(-\tau),\quad s>0. \qquad (*)$$

This makes sense for $\tau>0$ sufficiently small and the result does not depend on the choice of τ. It follows that $\Theta|W>$ is a ket and it is not a test ket, in general.

4.2. Example

Take a ket $|K>$ and a bra $<B|$. We define

$$|K><B|: (t,s)\mapsto |B>(t)\otimes|K>(s)$$

with $(|B>(t)\otimes|K>(s))f=(f,|B>(t))_X|K>(x),\ f\in X$. Then $|K><B|\in TT_{B_2(X),A\otimes I,I\otimes A}$.

4.3. Kernel theorem

All continuous linear mappings from $S_{X,A}$ into $T_{X,A}$ arise from the elements of $TT_{B_2(X),A\otimes I,I\otimes A}$ as described by (*).

In the next definition we introduce a linear subspace of $TT_{B_2(X),A\otimes I,I\otimes A}$.

4.4. Definition

$TS_{B_2(X),A\otimes I,I\otimes A}$ denotes the subspace of $TT_{B_2(X),A\otimes I,I\otimes A}$ which consists of all $\hat{K}$ with

$$\forall_{s>0}\exists_{t_s>0}\forall_{t>0}: \hat{L}(t,s)\in D(e^{t_sA}\otimes I)$$

(So for each fixed $s>0$ we can extend the function $t \mapsto \hat{L}(t,s)$ to the interval $(-t_s, \infty)$).

The action of $TS_{B_2(X),A\otimes I,I\otimes A}$ on the ket space $T_{X,A}$ is defined by

$$L|K>: s \mapsto \hat{L}(-\tau,s)(|K>(\tau)), \quad s>0, \qquad (**)$$

$\tau>0$ must be smaller than t_s. The definition of $L|K>$ does not depend on the choice of τ.

4.5. Example

Let $|K>$ be a ket and $<W|$ a test bra. We put

$$|K><W|: (t,s) \mapsto |W>(t)\otimes|K>(s), \quad t\geqslant -t_0, \quad s>0.$$

Then $|K><W| \in TS_{B_2(X),A\otimes I,I\otimes A}$.

4.6. Kernel theorem

All continuous linear mappings from $T_{X,A}$ into $T_{X,A}$ arise from the elements of $TS_{B_2(X),A\otimes I,I\otimes A}$ as described in (**).

Remark

The space $TS_{B_2(X),A\otimes I,I\otimes A}$ possesses the structure of an algebra,

$$\hat{L}_1 \cdot \hat{L}_2: (t,s) \mapsto L(-\sigma,s)L_2(t,\sigma)$$

with $s>0$, $t>-t_\sigma$ with $\sigma>0$ sufficiently small and dependent on s.

5. Dirac bases

We rephrase the definition of Dirac basis in terms of bras and kets.

5.1. Definition

Let (M,μ) be Federer measure space. A set $\{|x>|x\in M\}$ in the ket space $T_{X,A}$ is called a (canonical) Dirac basis if it possesses the following properties.

(a) For all $d\in\mathbb{D}$, $x\mapsto <x|v,d>(0)$ is a Borel function

(b) For all $d,\hat{d}\in\mathbb{D}$ the following relation is satisfied

$$\int_M <x|v,d>(0)<v,\hat{d}|x>(0)d\mu(x) = \delta_{d\hat{d}}.$$

(c) There exists a null set N such that for all $x\in M\setminus N$ all $d\in\mathbb{D}$ and all $n\in\mathbb{N}$

- $$<x|v,d>(0) = \lim_{r\downarrow 0} \mu(B(x,r))^{-1} \int_{B(x,r)} <y|v,d>(0)d\mu(y)$$

- $$<x|v,d>(0)|^2 = \lim_{r\downarrow 0} \mu(B(x,r))^{-1} \int_{B(x,r)} |<y|v,d>(0)|^2 d\mu(y)$$

- $$\sum_{p\in\mathbb{D}} \exp(-\frac{1}{n}\lambda_p)|<x|v,d>(0)|^2$$
$$= \lim_{r\downarrow 0} \mu(B(x,r))^{-1} \int_{B(x,r)} \sum_{p\in\mathbb{D}} e^{-\frac{1}{n}\lambda_p}|<y|v,d>(0)|^2 d\mu(y)$$

5.2. Theorem

Let $(|x>)_{x\in M}$ *be a Dirac basis,* $|W>$ *a test ket.*

(a) $\forall_{x\in M\setminus N}$: $<x|W>(0) = \lim_{r\downarrow 0} \mu(B(x,r))^{-1} \int_{B(x,r)} <y|W>(0)d\mu(y)$

(b) *For any test bra* $<V|$,

$$<V|W>(0) = \int_M <V|x>(0)<x|W>(0)d\mu(x)$$

For each $f\in X$ *define* $Uf\in L_2(M,\mu)$ *by*

$$Uf\colon x\mapsto \sum_{d\in\mathbf{D}} (f,v_d)<x|v,d>(0)$$

Then U *is a unitary operator from* X *onto a closed Hilbert subspace* Y *of* $L_2(M,\mu)$. *Define the positive self-adjoint operator* $\mathfrak{B}$ *in* Y *by* $\mathfrak{B}=U A U^*$.

Consider the following scheme

$$\begin{array}{ccccc} S_{X,A} & \hookrightarrow & X & \hookrightarrow & T_{X,A} \\ \downarrow U & & \downarrow U & & \downarrow U \\ S_{Y,B} & \hookrightarrow & Y & \hookrightarrow & T_{Y,B} \end{array}$$

Because of previous results, cf. Theorem 5.2, the elements of $S_{Y,B}$ can be regarded as genuine functions, which satisfy a relative differentiation result outside a fixed set $N\subseteq M$ with $\mu(N)=0$.

We define $\delta_y\in T_{Y,B}$ by

$$\delta_y(x,t) = \sum_{d\in\mathbf{D}} <v,d|y>(t)<x|v,d>(0)$$

i.e. $\delta_y=U|y>$.

So for each $x\in M$ and $t>0$ we have

$$<x|y>(t) = \delta_y(x,t). \tag{5.3}$$

Now for a test ket $|W>$ we have by 5.2.b.

$$<y|W>(0) = <y|e^{-\tau A}e^{\tau A}|W>(0)=\int_M <x|W>(-\tau)\delta_y(x,\tau)d\mu(x)$$

with $\tau>0$ sufficiently small. Thus $\delta_y\in T_{Y,B}$ acts as an evaluation functional on $S_{Y,B}$. Formula (5.3) interpretes the 'orthonormality relation' $<x|y> = \delta_y(x)$ as suggested by Dirac.

In general, the elements in $T_{Y,B}$ can be regarded as functions Φ of two variables x and t, $x\in M$, $t>0$ given by

$$\Phi(x,t) = <x|K>(t)$$

where $|K>$ is any ket. The space $T_{Y,B}$ contains the 'representatives' of the kets in $T_{X,A}$ corresponding to the representation induced by the Dirac basis $(|x>)_{x\in M}$. The next theorems give some formulae of Dirac and our interpretation of them.

5.4. Theorem

(a) $<B|K> = \int_M <B|x><x|K> d\mu(x)$

means

$$<B|K>(t) = \int_M <B|x>(\tau)<x|K>(t-\tau)d\mu(x)$$
$$= \int_M \overline{\Psi(x,\tau)}\Phi(x,t-\tau)d\mu(x), \quad t>0, \quad 0<\tau<t$$

where $U|B>=\Psi$ *and* $U|K>=\Phi$

(b) $|K> = \int_M |x><x|K> d\mu(x)$

means

$$|K>(t) = \int_M <x|K>(t-\tau)|x>(\tau)d\mu(x) = \int_M \Psi(x,t-\tau)|x>(\tau)d\mu(x),$$
$$t>0, \quad 0<\tau<t$$

(c) $<B| = \int_M <B|x>Mx|d\mu(x)$

means

$$<B|(t) = \int_M <B|x>(t-\tau)<x|(\tau)d\mu(x) = \int_M \Phi(x,t-\tau)<x|(\tau)d\mu(x),$$
$$t>0, \quad 0<\tau<t.$$

In his formalism Dirac suggests that the notion of matrix can be introduced for 'arbitrary' operators with respect to 'continuous' bases.

5.5. Definition

Let $\hat{\Theta} \in TT_{B_2(X), A\otimes I, I\otimes A}$. The matrix $[\Theta]$ of $\hat{\Theta}$ with respect to the Dirac basis $(|x>)_{x\in M}$ is defined by

$$[\Theta]_{xy}(t,s) = (\hat{\Theta}(t-\tau, s-\sigma), |y>(t)\otimes|x>(\sigma))_{B_2(X)}$$

where $t,s>0$ and $0<\sigma<s$, $0<\tau<t$. The definition of $[\Theta]$ does not depend on σ and τ.

The next theorem gives some matrix formulae of Dirac and our interpretation of them.

5.6. Theorem

(a) $\Theta = \int_{M\times M} [\Theta]_{xy}|x><y|d\mu(x)d\mu(y)$

means

$$\hat{\Theta}(t,s) = \int_{M\times M} [\Theta]_{xy}(\tau,\sigma)(|x><y|)(t-\tau,s-\sigma)d\mu(x)d\mu(y)$$

The integrals converge strongly in $X\otimes X$ *and do not depend on* τ, $0<\tau<t$ *and* σ, $0<\sigma<s$.

(b) *Let* $\hat{L}\in TS_{B_2(X);A\otimes I,I\otimes A}$. *Then its matrix* $[\mathfrak{L}]$ *with respect to the Dirac basis* $(|x>)_{x\in M}$ *has the property that for all* $s>0$ *there is* $t_s>0$ *such that for all* $(x,y)\in M\times M$ *the function* $t\mapsto[\mathfrak{L}]_{xy}(t,s)$ *extends to the interval* $(-t_s,\infty)$. *Let* $|K>$ *be any ket. Then*

$$\mathfrak{L}|K> = \int_{M\times M} [\mathfrak{L}]_{xy}|x><y|K>d\times(x)d\mu(y)$$

and

$$<x|\mathfrak{L}|K> = \int_M [\mathfrak{L}]_{xy}<y|K>d\mu(y)$$

means

$$\mathfrak{L}|K>:\ s\mapsto \int_{M\times M} [\mathfrak{L}]_{xy}(-\tau,\sigma)<y|F>(\tau)|x>(s-\sigma)d\mu(x)d\mu(y)$$

and

$$<x|\mathfrak{L}|K>:\ s\mapsto \int_{M\times M} [\mathfrak{L}]_{xy}(-\tau,s)<y|F>(\tau)\mu(y)$$

where $s>0$, $0<\sigma<s$ *and* $\tau>0$ *sufficiently small.*

(c) *Let* $\mathfrak{L}_1,\mathfrak{L}_2\in TS_{B_2(X);A\otimes I,I\otimes A}$. *Then Dirac's product formula*

$$[\mathfrak{L}_1\circ\mathfrak{L}_2]_{xy} = \int_M [\mathfrak{L}_1]_{xz}[\mathfrak{L}_2]_{zy}d\mu(z)$$

means

$$[\mathfrak{L}_1\circ\mathfrak{L}_2]_{xy}:\ (t,s)\mapsto \int_M [\mathfrak{L}_1]_{xz}(-\tau,s)[\mathfrak{L}_2]_{zy}(t,\tau)d\mu(z).$$

Here for each $s>0$ we must take $0<\tau<t_s^{(1)}$ and next $t>t_\tau^{(2)}$ may be taken.

For all remaining cases we refer to [EG1].

III. The Free Field Formalism

1. Hilbert space formulation

We start with a separable infinite dimensional Hilbert space X with orthonormal basis $(v_j)_{j\in\mathbb{N}}$. In X we define the positive self-adjoint operator A by

$$Af = \sum_{j=1}^{\infty} j(f,v_j)_X v_j, \quad f\in D(A),$$

where

$$D(A) = \{f\in X | \sum_{j=1}^{\infty} j^2 |(f,v_j)| < \infty\}.$$

E.g. we can take $X=L_2(\mathbb{R})$ and $A=\frac{1}{2}(-\frac{d^2}{dx^2}+x^2+1)$. Let $X(k)=X\otimes \cdots \otimes X$ (k-times) denote the k-fold Hilbert tensor product of X with inner product denoted by $(\cdot,\cdot)_{X(k)}$. For $f_1,\ldots,f_k\in X$, $f_1\otimes \cdots \otimes f_k \in X(k)$ is called the k-fold simple tensor product of $f_1,\ldots,f_k$. We note that

$$(f_1\otimes \cdots \otimes f_k,\, g_1\otimes \cdots \otimes g_k) = \prod_{l=1}^{k}(f_l,g_l)_X.$$

An orthonormal basis in $X(k)$ is given by the vectors $v_j(k)=v_{j_1}\otimes \cdots \otimes v_{j_k}$ with $j=(j_1,\ldots,j_k)\in\mathbb{N}^k$. In $X(k)$ introduce the positive self-adjoint operator $A(k)$ by

$$A(k)(v_{j_1}\otimes \cdots \otimes v_{j_k}) = |j|(v_{j_1}\otimes \cdots \otimes v_{j_k})$$

followed by linear and self-adjoint extension. Here $|j|=j_1+ \cdots +j_k$. We note that

$$e^{-tA(k)}(v_{j_1}\otimes\cdots\otimes v_{j_k}) = (e^{-tj_1}v_{j_1})\otimes\cdots\otimes(e^{-tj_k}v_{j_k}).$$

1.1. Lemma

Let $Y_0, Y_1, \cdots$ be Hilbert spaces and let $\mathcal{B}_0, \mathcal{B}_1, \ldots$ be self-adjoint operators in $Y_0, Y_1, \ldots$, respectively. Then the linear operator $\mathrm{diag}(\mathcal{B}_k)$ *in* $\oplus_{k=0}^{\infty} Y_k$ *defined by*

$$\mathrm{diag}(\mathcal{B}_k)\{f_k\} = \{\mathcal{B}_k f_k|, \quad \{f_k\}\in D(\mathrm{diag}(\mathcal{B}_k)).$$

where

$$D(\mathrm{diag}(\mathcal{B}_k)) = \{\{f_k\}\in \bigoplus_{k=0}^{\infty} D(\mathcal{B}_k)|\sum_{k=0}^{\infty}\|\mathcal{B}_k f_k\|_{Y_k}^2<\infty\},$$

is self-adjoint.

We introduce the Fock space F,

$$F = \bigoplus_{k=0}^{\infty} X(k)$$

with $X(0) = \mathbf{C}$ and $X(1)=X$, and in F the self-adjoint operator $\mathcal{H}$.

$$\mathcal{H} = \mathrm{diag}(A(k))$$

where we set $A(o)=0$ and $A(t)=A$. The operator $\mathcal{H}$ has discrete spectrum with eigenvalues $0,1,2,\ldots$ and corresponding multiplicities $m_0=1$, $m_N=2^{N-1}$, $N=1,2,\ldots$.

1.2. Proposition

Let $k\in\mathbf{N}$, $g\in X$. The operator $a_k(g)$: $X(k)\to X(k-1)$ is defined by

$$a_k(g)f(k) = \sum_{j\in\mathbf{N}^{k-1}}\sum_{j_1\in\mathbf{N}}(v_{j_1},g)_X(f(k),v_{j_1j}(k))_{X(k)}v_j(k-1)$$

where we set

$$v_{j_1j}(k) = v_{j_1}\otimes v_j(k-1) = v_{j_1}\otimes(v_{j_2}\otimes\cdots\otimes v_{j_k}).$$

Then $a_k(g)$: $X(k)\to X(k-1)$ is continuous with $\|a_k(g)\|=\|g\|_X$. In addition for all $t>0$

$$a_k(e^{-tA}g) = e^{tA(k-1)}a_k(g)e^{-tA(k)}.$$

Let $c_k(g)=a_k(g)^*$. It is clear that $c_k(g)$ is a continuous linear operator from $X(k-1)$ into $X(k)$ with $\|c_k(g)\|=\|g\|_X$. We observe that

$$c_k(g)f(k-1) = g\otimes f(k-1), \quad f(k-1)\in X(k-1).$$

1.3. Lemma

For each $k\in\mathbf{N}$ the mappings a_k: $X\to\mathcal{B}(X(k),X(k-1))$ and c_k: $X\to\mathcal{B}(X(k-1),X(k))$ are continuous. In particular we have

$$a_k(g) = \sum_{l=1}^{\infty} (v_l, g)_X a_k(v_l)$$

and

$$c_k(g) = \sum_{l=1}^{\infty} (g, v_l)_X c_k(v_l).$$

The mapping a_k is anti-linear and the mapping c_k is linear.

In the Fock space $F = \oplus_{k=0}^{\infty} X(k)$ we introduce the dense subspaces D_α, $\alpha > 0$.

$$D_\alpha = \{\{f(k)\} \in F \mid \sum_{k=0}^{\infty} k^\alpha \|f(k)\|_{X(k)}^2 < \infty\}.$$

1.4. Definition

On D_1 we introduce the linear operator $a(g)$ by

$$a(g)\{f(k)\} = \{\sqrt{k+1}\, a_{k+1}(g) f(k+1)\}.$$

So $a(g)$ is represented by the operator matrix

$$a(g) = \begin{bmatrix} 0 & \sqrt{1}\; a_1(g) & & & \emptyset \\ & 0 & \sqrt{2}\; a_2(g) & & \\ & & 0 & \sqrt{3}\; a_3(g) & \\ & \emptyset & & \ddots & \ddots \end{bmatrix}$$

(Observe that $a(g)$ is an unbounded linear operator.)

1.5. Lemma

Set $c(g) := a(g)^*$. Then $D(c(g)) = D_1 = D(a(g))$ and $c(g)^* = a(g)^{**} = a(g)$. So both $a(g)$ and $c(g)$ are closed linear operators in F. On D_1 we have

$$c(g)\{f(k)\} = \{\sqrt{k}\, c_k(g) f(k-1)\}.$$

So $c(g)$ is represented by the operator matrix

$$c(g) = \begin{bmatrix} 0 & & & & \emptyset \\ \sqrt{1}\; c_1(g) & 0 & & & \\ & \sqrt{2}\; c_2(g) & 0 & & \\ & & \sqrt{3}\; c_3(g) & 0 & \\ & \emptyset & & \ddots & \ddots \end{bmatrix} \quad \square$$

Remark

The operators $\phi(g) = \dfrac{a(g)+c(g)}{\sqrt{2}}$ and $\phi+(g) = \dfrac{a(g)-c(g)}{i\sqrt{2}}$ are self-adjoint in F.

1.6. Lemma

(a) $a(e^{-tA}g)=e^{t\mathcal{K}}a(g)e^{-t\mathcal{K}}$, $t>0$, $g\in X$

(b) $c(e^{-tA}g)=e^{-t\mathcal{K}}a(g)e^{t\mathcal{K}}$, $t>0$, $g\in X$

(c) *For each* $\{f(k)\}\in D_1$

$$a(g)\{f(k)\} = \sum_{l=1}^{\infty}(v_l,g)_X a(v_l)\{f(k)\}$$

and

$$c(g)\{f(k)\} = \sum_{l=1}^{\infty}(g_l,v)_X c(v_l)\{f(k)\}$$

where convergence is in the norm of F. □

The next step in our construction is the introduction of the symmetrization projection $P^{(+)}$ and anti-symmetrization projection $P^{(-)}$.

Let $\mathbf{P}_k$ denote the permutation group of order k. For each permutation $\sigma\in\mathbf{P}_k$ we introduce the unitary operator $\hat{\sigma}$ on $X(k)$ by

$$\hat{\sigma}(f(k)) = \sum_{j\in\mathbf{N}^k}(f(k),v_j(k))_{X(k)}v_{\sigma(j)}(k)$$

where $\sigma(j)=(j_{\sigma(1)},\ldots,j_{\sigma(k)})$. Now we put

$$\mathcal{P}^{(+)}(k) = \frac{1}{k!}\sum_{\sigma\in\mathbf{P}_k}\hat{\sigma}.$$

Then $\mathcal{P}^{(+)}(k)$ is an orthogonal projection in $X(k)$. Its range is denoted by $X^{(+)}(k)$ and called the k-fold symmetric tensor product of X. Further, we put $A^{(+)}(k)-\mathcal{P}^{(+)}(k)A(k)\mathcal{P}^{(+)}(k)$. Similarly we introduce the projection $\mathcal{P}^{(-)}(k)$ in $X(k)$ by

$$\mathcal{P}^{(-)}(k) = \frac{1}{k!}\sum_{\sigma\in\mathbf{P}_k}\epsilon(\sigma)\hat{\sigma}.$$

where $\epsilon(\sigma)=1$ if σ is even and $\epsilon(\sigma)=-1$ if σ is odd. The range of $\mathcal{P}^{(-)}(k)$ is denoted by $X^{(-)}(k)$ and called the k-fold anti-symmetric tensor product of X. We set $A^{(-)}(k)=\mathcal{P}^{(-)}(k)A(k)\mathcal{P}^{(-)}(k)$.

1.7. Definition

The orthogonal projections $\mathcal{P}^{(+)}$ and $\mathcal{P}^{(-)}$ in F are defined by

$$\mathcal{P}^{(\pm)} = \mathrm{diag}(\mathcal{P}^{(\pm)}(k)).$$

We set $F^{(\pm)}=\mathcal{P}^{(\pm)}(F)$. $F^{(+)}$ is called the Boson Fock space and $F^{(-)}$ the Fermion Fock space. Further, we introduce the positive self-adjoint operator $\mathcal{K}^{(\pm)}$

in $F^{(\pm)}$ by

$$\mathcal{H}^{(\pm)} = \mathcal{P}^{(\pm)}\mathcal{H}\mathcal{P}^{(\pm)}.$$

We note that $\mathcal{H}\mathcal{P}^{(\pm)} = \mathcal{P}^{(\pm)}\mathcal{H}$.

The operator $\mathcal{H}^{(+)}$ has discrete spectrum with eigenvalues $N=0,1,2,\ldots$ and corresponding multiplicity $m_N^{(+)}$. Here $m_N^{(+)}$, $N=1,2,\ldots$ equals the number of decompositions of N into integer summands without regard to order. The asymptotics of $m_N^{(+)}$ is given by $m_N^{(+)} \sim \frac{1}{4N\sqrt{3}} e^{\pi\sqrt{\frac{2}{3}N}}$. So for all $t>0$ the operator $e^{-t\mathcal{H}^{(+)}}$ is Hilbert-Schmidt. The operator $\mathcal{H}^{(-)}$ has discrete spectrum with eigenvalues $N=0,1,2,\ldots$ and multiplicities $m_N^{(-)}$. Here $m_N^{(-)}$, $N=1,2,\ldots$, equals the number of decompositions of N into distinct integer summands without regard to order. In this case we have $m_N^{(-)} \sim \sqrt[4]{3N^3}\, e^{\pi\sqrt{\frac{1}{3}N}}$. So the operator $e^{-t\mathcal{H}^{(-)}}$ is Hilbert-Schmidt for all $t>0$.

1.8. Definition

Let $g\in X$. On $D_1^{(\pm)} = \mathcal{P}^{(\pm)}(D_1)$ we introduce the operators

$$a^{(\pm)}(g) = \mathcal{P}^{(\pm)}a(g)\mathcal{P}^{(\pm)}$$

$$c^{(\pm)}(g) = \mathcal{P}^{(\pm)}c(g)\mathcal{P}^{(\pm)}$$

We have $a^{(\pm)}(g)^* = c^{(\pm)}(g)$ and $c^{(\pm)}(g)^* = a^{(\pm)}(g)$.

Remark

- The operator $a^{(\pm)}(g)$, $g\in X$, may be called annihilation operators. The operators $c^{(\pm)}(g)$, $g\in X$, may be called creation operators.
- Observe that $a^{(\pm)}(g) = a(g)\mathcal{P}^{(\pm)}$ and $c^{(\pm)}(g) = \mathcal{P}^{(\pm)}c(g)$.

The following commutation relations and expansions are valid.

1.9. Theorem

Let $g,h\in X$. Then on $D_2^{(\pm)} = \mathcal{P}^{(\pm)}(D_2)$

$$[a^{(\pm)}(g)\ a^{(\pm)}(h)]_{\mp} = 0$$

$$[c^{(\pm)}(g)\ c^{(\pm)}(h)]_{\mp} = 0$$

$$[a^{(\pm)}(g)\ c^{(\pm)}(h)]_{\mp} = (h,g)_X I^{(\pm)}.$$

Further, from Lemma 1.6 we obtain the expansions

$$a^{(\pm)}(g) = \sum_{l=1}^{\infty}(v_l,g)_X a^{(\pm)}(v_l)$$

$$c^{\pm}(g) = \sum_{l=1}^{\infty}(g,v_l)_X c^{(\pm)}(v_l)$$

where convergence takes place pointwise on $D_1^{(\pm)}$. □

Remark
The operators $\{a^{(+)}(g), c^{(+)}(g), I^+\}$ establish a representation of the Heisenberg algebra in infinitely many variables.

2. Some mathematical tools
We introduce the trajectory space $T_{X(k),A(k)}$ and the analyticy space $S_{X(k),A(k)}$. The kets in $T_{X(k),A(k)}$ are denoted by $|\Phi;k>$. So $|\Phi;k>$ is an $X(k)$-valued function on $(0,\infty)$ with the property

$$|\Phi;k>(t+\tau) = e^{-\tau A(k)}|\Phi;k>(t), \quad t,\tau>0.$$

The elements of $T_{X(k),A(k)}$ are called k-particles kets. A ket $|\Omega;k>$ belongs $S_{X(k),A(k)}$ if there exists a ket $|\Phi;k>$ and $t>0$ such that

$$|\Omega;k> = e^{-\tau A(k)}|\Phi;k>.$$

The kets in $S_{X(k),A(k)}$ are called k-particles test kets. Similarly, we introduce the spaces $T_{X(k)',A(k)'}$ and $S_{X(k)',A(k)'}$. The elements of $S_{X(k)',A(k)'}$ are denoted by $<\Phi;k|$. They are called k-particles bras. By $|j;k>$ we denote the k-particles ket

$$|j;k>: t \mapsto e^{-t|j|}(v_{j_1} \otimes \cdots \otimes v_{j_k})$$

and, correspondingly, by $<j,k|$ the k-particles bra

$$|j;k>: t \mapsto e^{-t|j|}(v'_{j_1} \otimes \cdots \otimes v'_{j_k})$$

2.1. Proposition
(a) *Let* $|K_1>, \ldots, |K_k>$ *be one-particle kets. Then*

$$t \mapsto \sum_{j \in \mathbf{H}^k} <j_1|K_1>(0) \cdots <j_k|K_k>(0)|j,k>(t), \quad t>0,$$

is a member of $T_{X(k),A(k)}$. *This k-particles ket is denoted by* $|K_1> \cdots |K_k>$. *Observe that for all* $t>0$

$$(|K_1> \cdots |K_k>)(t) = |K_1>(t) \otimes \cdots \otimes |K_k>(t)$$

(b) *Let* $|W_1>, \ldots, |W_k>$ *be one-particle test kets. Then* $|W_1>|W_2> \cdots |W_k>$ *is a member of* $S_{X(k),A(k)}$.

(c) *Let* $<B_1|, \ldots, <B_k|$ *be one particles bras. Then*

$$t \mapsto \sum_{j \in \mathbf{H}^k} <B_1|j_1>(0) \cdots <B_k|j_k>(0)<j,k|(t), \quad t>0,$$

is a member of $T_{X(k)',A(k)'}$. *This k-particles bra is denoted by* $|B_1>, \ldots, |B_2|<B_1|$. *for all* $t>0$

$$(<B_k| \cdots <B_1|)(t) = <B_1|(t) \otimes \cdots \otimes <B_k|(t)$$

(d) *Let* $<V_1|, \ldots, <V_k|$ *be one-particle test bras. Then* $<V_k| \cdots$

$<V_2|<V_1|$ *is a member of* $S_{X(k)',A(k)'}$.

2.2. Definition

The Fock trajectory space $T_{F,\mathcal{H}}$ consists of all mapping Φ from $(0,\infty)$ into F with the property that for all $t>0$ and $\tau>0$.

$$\Phi(t+\tau) = e^{-\tau\mathcal{H}}\Phi(t).$$

The Fock analyticity space $S_{F,\mathcal{H}}$ consists of all $\Omega \in T_{F,\mathcal{H}}$ for which $\Phi \in T_{F,\mathcal{H}}$ and $\tau>0$ exist such that $\Omega = e^{-\tau\mathcal{H}}\Phi$.

2.3. Lemma

(a) $\Phi \in T_{F,\mathcal{H}}$ *iff there exist* $|\Phi;k> \in T_{X(k),A(k)}$, $k=0,1,2,\ldots$ *such that* $\forall_{t>0}$: $\{|\Phi,k>(t)\} \in F$ *and*

$$\Phi(t) = \{|\Phi;k>(t)\}, \quad t>0.$$

(b) $\Omega \in T_{F,\mathcal{H}}$ *iff there exist* $|\Omega;k> \in S_{X(k),A(k)}$, $k=0,1,2,\ldots$ *such that* $\exists_{\tau>0}$: $\{|\Omega,k>(-\tau)\} \in F$ *and*

$$\Omega(t) = \{|\Omega;k>(t)\}, \quad t>-\tau. \quad \square$$

Remarks

- The same definitions and results apply with replaced by F' and $\mathcal{H}$ replaced by $\mathcal{H}'$.
- The space $T_{F,\mathcal{H}}$ $(T_{F',\mathcal{H}'})$ is Montel but not nuclear.

For any bra $<B|$ and any ket $|K>$ we introduce the linear mappings $N(<B|)$ and $M(|K>)$.

2.4. Definition

(a) Let $<B|$ be any bra. The mapping $N(<B|)$ from $S_{F,\mathcal{H}}$ into $S_{F,\mathcal{H}}$ is defined by

$$N(<B|)\{|\Omega;k>\}: t \mapsto e^{-(t+\tau)\mathcal{H}}a(|B>(\tau))\{(W;k>(-\tau)\}.$$

Because of 1.6.a this definition does not depend on the choice of $\tau>0$ sufficiently small.

(b) In addition, let $<W|$ be a test bra. Then the mapping $N(<W|)$ of (a) extends to a linear mapping from $T_{F,\mathcal{H}}$ into $T_{F,\mathcal{H}}$. We have

$$N(<W|)\{|\Phi;k>\}: t \mapsto e^{-(t+\tau)\mathcal{H}}a(|W>(-\tau))\{(\Phi;k>(\tau)\}$$

where $0<\tau<t$ has to be taken sufficiently small.

(c) Let $|K>$ be any ket. Then the linear mapping $M(|>)$ from $T_{F,\mathcal{H}}$ into $T_{F,\mathcal{H}}$ is defined by

$$M(|K)\{|\Phi;k>\}: t \mapsto e^{-(t+\tau)\mathcal{H}}c(|K>(\tau))\{(\Phi;k>(\tau)\}.$$

where we have to take $0<\tau<t$. By 1.6.a the definition does not depend on τ.

(d) In addition, let $|W>$ be a test ket. Then $M(|W>)$ maps $S_{F,\mathcal{H}}$ into $S_{F,\mathcal{H}}$.

For $\{\Omega;k>\}\in S_{F,\mathcal{K}}$ we have

$$M(|W>)\{|\Omega;k>\}:\ t\mapsto e^{-(t+\tau)\mathcal{K}}c(|W>(-\tau))\{(\Omega;k>(-\tau)\}$$

where $\tau>0$ has to be taken sufficiently small.

Now let $(|\xi>)_{\xi\in M}$ be a Dirac basis. We want to interpret the following heuristic formulae,

$$N(<B|) = \int_M <B|\xi>N(<\xi|)d\mu(\xi)$$

and

$$M(|K>) = \int_M <\xi|K>M(<\xi|)d\mu(\xi).$$

which extend the formulae 1.6.b.

2.5. Lemma

(a) $N(<B|) = \int_M <B|\xi>N(<\xi|)d\mu(\xi)$ *can be interpreted as*

$$\forall_{t>0}\forall_{\tau,0<\tau<t}:\ e^{\tau\mathcal{K}}N(<B|)e^{-t\mathcal{K}} = (a(|B>(\tau))e^{-(t-\tau)\mathcal{K}})$$
$$= \int_M <B|\xi>(\tilde{\tau})e^{(\tau-\tilde{\tau})\mathcal{K}}N(<\xi|)e^{-(t-\tilde{\tau})\mathcal{K}}d\mu(\xi)$$

where $0<\tilde{\tau}<\tau$ *can be arbitrary. For each* $t,\tau,\tilde{\tau}$ *with* $0<\tilde{\tau}<\tau<t$ *the operator valued function*

$$\xi\mapsto <B|\xi>(\tilde{\tau})e^{(\tau-\tilde{\tau})\mathcal{K}}N(<\xi|)e^{-(t-\tilde{\tau})\mathcal{K}}$$

is strongly μ*-integrable w.r.t.* $\mathcal{B}(F)$.

(b) $M(<K|) = \int_M <\xi|K>M(<\xi|)d\mu(\xi)$ can be interpreted as

$$\forall_{t>0}\forall_{\tau,0<\tau<t}:\ e^{-\tau\mathcal{K}}M(<K|)e^{t\mathcal{K}} = e^{-(t-\tau)\mathcal{K}}(c(|K>(\tau))$$
$$= \int_M <\xi|K>(\tilde{\tau})e^{(\tau-\tilde{\tau})\mathcal{K}}M(|\xi>)e^{(\tau-\tilde{\tau})\mathcal{K}}d\mu(\xi)$$

where $0<\tilde{\tau}<\tau$ can be arbitrary. For each $t,\tau,\tilde{\tau}$ with $0<\tilde{\tau}<\tau<t$ the operator valued function

$$\xi\mapsto <\xi|K>(\tilde{\tau})e^{-(t-\tilde{\tau})\mathcal{K}}M(|\xi>)e^{(\tau-\tilde{\tau})\mathcal{K}}$$

is strongly μ-integrable w.r.t. $\mathcal{B}(F)$. □

2.6. Definition

Let $\mathfrak{L}$ be a linear mapping from the one-particle ket space $T_{X,A}$ into $T_{X,A}$. By the integral expression

$$\int_M M(\mathfrak{L}|\xi>)N(<\xi|)d\mu(\xi)$$

we mean the linear mapping from $S_{F,\mathcal{K}}$ into $T_{F,\mathcal{K}}$ defined by

$$(\int_M M(\wp|\xi>)N(<\xi|)d\mu(\xi)\{|\Omega;k>\}:$$

$$t\mapsto(\int_M e^{-t\mathcal{H}}(\wp|\xi>)N(<\xi|))e^{-\tau\mathcal{H}}d\mu(\xi))\{|\Omega;k>(-\tau)\}.$$

Here $\tau>0$ has to be taken sufficiently small.

Remark
For all $t,\tau>0$ the $\mathfrak{B}$-valued function

$$\xi\mapsto e^{-t\mathcal{H}}M(\wp|\xi>)N(<\xi|)e^{-\tau\mathcal{H}},\quad \xi\in M,$$

is strongly integrable for each t, $\tau>0$.

2.7. Theorem
The equality

$$\int_M M(\wp|\xi>)N(<\xi|d\mu(\xi) = \int_M\int_M M(|\xi'>)<\xi'|\wp|\xi>N(<\xi|)d\mu(\xi)d\mu(\xi')$$

admits the following interpretation:

$$\forall_{t>0}\forall_{\tau,0<\tau<t}: e^{-t\mathcal{H}}(\int_M M(\wp|\xi>)N(<\xi|d\mu(\xi))e^{-t\mathcal{H}}=$$

$$=\iint_{M\times M}<\xi'|\wp|\xi>(t-\tau)e^{-\tau\mathcal{H}}M(|\xi'>)e^{\tau\mathcal{H}}N(<\xi|)e^{-t\mathcal{H}}d\mu(\xi)d\mu(\xi').\quad\square$$

Remark
For $t>0$ and τ, $0<\tau<t$, the $B(F)$-valued function

$$(\xi,\xi')\mapsto <\xi'|\wp|\xi>(t-\tau)e^{-\tau\mathcal{H}}M(|\xi'>)e^{\tau\mathcal{H}}N(<\xi|)e^{-t\mathcal{H}}$$

is strongly $\mu\otimes\mu$-integrable.

3. Trajectory space formulation

Starting from the projections $\mathcal{P}^{(\pm)}(k)$ of the first section we introduce the spaces $T_{X^{(\pm)}(k),A^{(\pm)}(k)}$ and $S_{X^{(\pm)}(k),A^{(\pm)}(k)}$. The spaces with plus sign contain the k-bosons (test) kets; the spaces with minus sign the k-fermions (test) kets. The projections $\mathcal{P}^{(\pm)}$: $F\to F^{(+)}$ induce the Boson-Fock trajectory space $T_{F^{(+)},\mathcal{H}^{(+)}}$ and the Fermion-Fock trajectory space $T_{F^{(-)},\mathcal{H}^{(-)}}$, respectively. The space $T_{F^{(+)},\mathcal{H}^{(+)}}$ contains the Boson field kets and the space $T_{F^{(-)},\mathcal{H}^{(-)}}$ the Fermion field kets. As observed in Section III.1 the operators $e^{-t\mathcal{H}^{(\pm)}}$, $t>0$, are Hilbert-Schmidt, whence the spaces $T_{F^{(\pm)},\mathcal{H}^{(\pm)}}$ are nuclear.

3.1. Definition
(a) Let $<B|$ be a one-particle bra. We introduce the field operator

$$N^{(\pm)}(<B|) = \mathcal{P}^{(\pm)}N(<B|)\mathcal{P}^{(\pm)}.$$

Observe that $e^{t\mathcal{H}^{(\pm)}}N^{(\pm)}(<B|)e^{-t\mathcal{H}^{(\pm)}}=a^{(\pm)}(|B>(t))$, $t>0$.

(b) Let $|K>$ be a one-particle ket. We introduce the field operator

$$M^{(\pm)}(|K>) = \mathcal{P}^{(\pm)}M(|K>)P^{(\pm)}$$

Observe that $e^{-t\mathcal{H}^{(\pm)}}M^{(\pm)}(|K>)e^{t\mathcal{H}^{(\pm)}} = c^{(\pm)}(|K>(t))$. The field operators $N^{(\pm)}(<B|)$ are called annihilation operators. They act from $S_{F^{(\pm)},\mathcal{H}^{(\pm)}}$ into $S_{F^{\pm)},\mathcal{H}^{(\pm)}}$. If $<W|$ is a test bra then $N^{(\pm)}(<W|)$ extends to a linear mapping from $T_{F^{(\pm)},\mathcal{H}^{(\pm)}}$ into $T_{F^{(\pm)},\mathcal{H}^{(\pm)}}$.

The field operators $M^{(\pm)}(|K>)$ are called creation operators. These operators act from $T_{F^{(\pm)},\mathcal{H}^{(\pm)}}$ into $T_{F^{\pm},\mathcal{H}^{(\pm)}}$. If $<W|$ is a test ket, then $M^{(\pm)}(|W>)$ maps $S_{F^{(\pm)},\mathcal{H}^{(\pm)}}$ into $S_{F^{\pm},\mathcal{H}^{(\pm)}}$.

In the next theorem we present some heuristic formulae, used in the free field formalism, together with our interpretation of them.

3.2. Theorem

Let $(|\xi>)_{\xi\in M}$ *be a Dirac basis.*

(a) *The expansion*

$$N^{(\pm)}(<B|) = \int_M <B|\xi>N^{(\pm)}(<\xi|)d\mu(\xi)$$

means

$$\forall_{t>0}\forall_{\tau,0<\tau<t}\forall_{\tilde{\tau},0<\tilde{\tau}<t}:\ e^{\tau\mathcal{H}^{(\pm)}}N^{(\pm)}(<B|)e^{-t\mathcal{H}^{(\pm)}} =$$
$$= \int_M <B|\xi>(\tilde{\tau})e^{-(\tau-\tilde{\tau})\mathcal{H}^{(\pm)}}N^{(\pm)}(<\xi|)e^{-(\tau-\tilde{\tau})\mathcal{H}^{(\pm)}}d\mu(\xi)$$

Cf. Lemma 2.5.a.

(b) *The expansion*

$$M^{(\pm)}(|K>) = \int_M <\xi|K>M^{(\pm)}(|\xi>)d\mu(\xi)$$

means

$$\forall_{t>0}\forall_{\tau,0<\tau<t}\forall_{\tilde{\tau},0<\tilde{\tau}<t}:\ e^{-t\mathcal{H}^{(\pm)}}M^{(\pm)}(|K>)e^{\tau\mathcal{H}^{(\pm)}} =$$
$$= \int_M <\xi|K>(\tilde{\tau})e^{-(t-\tilde{\tau})\mathcal{H}^{(\pm)}}M^{(\pm)}(|\xi>)e^{(\tau-\tilde{\tau})\mathcal{H}^{(\pm)}}d\mu(\xi).$$

Cf. Lemma 2.5.b. □

For each pair of bras $(<B_1|, <B_2|)$ the (anti-) commutator

$$[N^{(\pm)}(<B_1|), N^{(\pm)}(<B_2|)]_{\mp}$$

is a well defined linear mapping from $S_{F^{(\pm)},\mathcal{H}^{(\pm)}}$ into $S_{F^{(\pm)},\mathcal{H}^{(\pm)}}$. Similarly, for each pair of kets $(|K_1>, |K_2>)$ the (anti-) commutator

$$[M^{(\pm)}(|K_1>), M^{(\pm)}(|K_2>)]_{\mp}$$

is a well defined linear mapping from $T_{F^{(\pm)},\mathcal{H}^{(\pm)}}$ into $T_{F^{(\pm)},\mathcal{H}^{(\pm)}}$. Also, for each ket $|K>$ and each test bra $<W|$ the (anti-) commutator

$$[N^{(\pm)}(<W|), M^{(\pm)}(|K>)]_{\mp}$$

is a well defined linear mapping from $T_{F^{(\pm)},\mathcal{K}^{(\pm)}}$ into $T_{F^{(\pm)},\mathcal{K}^{(\pm)}}$.

3.3. Theorem

Let $<B_1|$, $<B_2|$ be one-particle bras, $|K>$, $|K_1>$, $|K_2>$ be one-particle kets and $<W|$ a one-particle test bra.

(a) $[N^{(\pm)}(<B_1|), N^{(\pm)}(<B_2|)]_{\mp} = 0$

(b) $[M^{(\pm)}(|K_1>), M^{(\pm)}(|K_2>)]_{\mp} = 0$

(c) $[N^{(\pm)}(<W|), M^{(\pm)}(|K>)]_{\mp} = <W|K>(0)I^{(\pm)}$.

It follows from 3.3.c that for any bra $<B|$ and any ket $|K>$.

$$[N^{(\pm)}(<B|e^{tA}), M^{(\pm)}|K>]_{\mp} = <B|K>(t)I^{(\pm)}, \quad t>0.$$

3.4. Definition

The heuristic (anti-) commutator expression

$$[N^{(\pm)}(<B|), M^{(\pm)}(|K>)]_{\mp}$$

denotes the operator valued function

$$t \mapsto [N^{\pm}(<B|e^{-tA}), M^{\pm}(|K>)]_{\mp}, \quad t>0.$$

Thus we arrive at the continuum version of the CAR and CCR.

3.5. Corollary

Let $(\xi>)_{\xi\in M}$ be a Dirac basis. Then for all $\xi,\eta\in M$

$$[N^{(\pm)}(<\xi|), N^{(\pm)}(<\eta|)]_{\mp} = 0$$

$$[M^{(\pm)}(|\xi>), M^{(\pm)}(|\eta>)]_{\mp} = 0$$

$$[N^{(\pm)}(<\xi|), M^{(\pm)}(|\eta>)]_{\mp} = <\xi|\eta>I^{(\pm)} = \delta_\eta(\xi)I^{(\pm)} \quad \square$$

Finally we introduce the notion of second quantization. In Theorem 2.7 we have given our mathematical interpretation of the expression

$$\int_M\int_M d\mu(\xi)d\mu(\xi')M(|\xi')<\xi'|\mathfrak{L}|\xi>N(<\xi|)$$

where $\mathfrak{L}$ is a linear mapping from $T_{X,A}$ into $T_{X,A}$. Multiplying both sides by $\mathcal{P}^{(\pm)}$ yields

$$\begin{aligned}\Omega^{(\pm)}(\mathfrak{L}) &= \mathcal{P}^{(\pm)}\Big(\int_M\int_M d\mu(\xi)d\mu(\xi')M(|\xi')<\xi'|\mathfrak{L}|\xi>N(<\xi|)\Big)\mathcal{P}^{(\pm)} \\ &= \int_M\int_M d\mu(\xi)d\mu(\xi')M^{(\pm)}(|\xi')<\xi'|\mathfrak{L}|\xi>N^{(\pm)}(<\xi|).\end{aligned}$$

The operator $\Omega^{(\pm)}(\mathfrak{L})$: $S_{F^{(\pm)},\mathcal{K}^{(\pm)}} \to T_{F^{(\pm)},\mathcal{K}^{(\pm)}}$ is called the second quantization of $\mathfrak{L}$. In particular, $\Omega^{(\pm)}(\mathfrak{L})$ is the so-called number operator with the property,

$$\Omega^{(\pm)}(I)\mathcal{P}^{(\pm)}\{|\Phi;k_0>\delta_{k_0k}\} = k_0\mathcal{P}^{(\pm)}\{|\Phi;k_0>\delta_{kk_0}\}.$$

In litarature one finds the suggestion that 'every' operator can be expanded in annihilation and creation operators. Two types of expansions seem fashionable. One with a projection on the vacuum state in the middle, with the annihilation operators on the right hand side and the creation operators on the left hand side, and one such an expansion without this projection. The latter is called the normal form. The first of these claims can be proved in the present context, where we can even make an expansion involving an arbitrary Dirac basis. We present a sketch of the proof.

Starting from the Dirac basis $(|x>)_{x\in M}$ we obtain a Dirac basis in $T_{F^{(\pm)},\mathcal{K}^{(\pm)}}$ given by

$$\{\mathcal{G}^{\pm}\{|x;k_0>\delta_{kk_0}\}|x\in M^{k_0},\quad k_0\in\mathbb{N}\cup\{0\}\}$$

with

$$|x;k_0> = |x_1>|x_2>\cdots|x_{k_0}>,\quad x=(x_1,\ldots,x_{k_0})\in M^{k_0}.$$

The underlying Federer measure space for this Dirac basis is $M(\infty)=\bigcup_{k=0}^{\infty}M^k$, i.e. the disjoint union of all k-fold Cartesian products of the measure space M. Now let $\Xi^{(\pm)}: S_{F^{(\pm)},\mathcal{K}^{(\pm)}}\to T_{F^{(\pm)},\mathcal{K}^{(\pm)}}$ be a continuous linear mapping. Because of the nuclearity of these there exists $\hat{\Xi}^{(\pm)}\in TT_{\mathcal{G}^{(\pm)}\otimes F^{(\pm)};\mathcal{K}^{(\pm)}\otimes I^{(\pm)},I^{(\pm)}\otimes\mathcal{K}^{(\pm)}}$ which represents $\Xi^{(\pm)}$ as described in Kernel Theorem II.4.3. Now with the matrix, cf. Definition II.5.5.

$$a_{kl}^{(\mp)}(x_1,\ldots,x_k,y_1,\ldots,y_l;t,s) = [\Xi^{(\pm)}]_{(x,k,)(y,l)}(t,s)$$
$$=(\hat{\Xi}^{(\pm)}(t-\tau,s-\sigma),\mathcal{G}^{(\pm)}\{|y;l>(\tau)\delta_{ll'}\}\otimes\mathcal{G}^{(\pm)}\{|x,k>(\sigma)\delta_{kk'}\}$$

we have the expansion result

$$e^{-t\mathcal{K}^{(\pm)}}\Xi^{(\pm)}e^{-s\mathcal{K}^{(\pm)}} = \int_{M(\infty)}\int_{M(\infty)} a_{kl}^{(\mp)}(x_1,\ldots,x_k,y_1,\ldots,y_l;\tau,\sigma)$$
$$\cdot(\mathcal{G}^{(\pm)}\{|y;l>\delta_{ll'}\}\{<x;k|\delta_{kk'}\}\mathcal{G}^{(\pm)})(t-\tau,s-\sigma)d\mu_\infty(x)d\mu_\infty(y).$$

In the latter integral we insert the relations

$$\mathcal{G}^{(\pm)}\{|y;l>\delta_{ll'}\} = M^{(\pm)}(|y_1>)\cdots M^{(\pm)}(y_l)|0>$$
$$\{<x;k|\delta_{kk'}\}\mathcal{G}^{(\pm)} = <0|N^{(\pm)}(<x_1|)\cdots N^{(\pm)}(<x_k|).$$

Then we ultimately arrive at the expression

$$e^{-t\mathcal{K}^{(\pm)}}\Xi^{(\pm)}e^{-s\mathcal{K}^{(\pm)}} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\underbrace{\int\cdots\int}_{k}\underbrace{\int\cdots\int}_{k} a_{kl}^{(\mp)}(x_1,\ldots,x_k,y_1,\ldots,y_l;\tau,\sigma)\cdot$$
$$\cdot e^{-(t-\tau)\mathcal{K}^{(\pm)}}M^{(\pm)}(|y_1>)\cdots M^{(\pm)}(|y_l>)|0><0|N^{(\pm)}(<x_1|)\cdots N^{(\pm)}(<x_k|)$$
$$\cdot e^{-(s-\sigma)\mathcal{K}^{(\pm)}}d\mu(x_1)\cdots d\mu(y_1)\cdots d\mu(y_l).$$

This is a mathematical expression of the familiar heuristic expression, which can be obtained by taking t,τ,s and σ all equal to zero.

In literature, e.g. [Sh] p. 22, one also encounters this type of expansion expressions without the projection on the vacuum state $|0><0|$ in the middle.

Perhaps convergence of such expressions involving an arbitrary Dirac basis can be dealt with in a similar way. At this moment we do not know whether each operator can be represented in this so-called normal form. In literature such statements can be found without proof or reference.

[Di] DIRAC, P.A., The principles of quantum mechanics. Fourth edition, Clarendon press. Oxford, 1958.

[EG1] EIJNDHOVEN, S.J.L. VAN, J. DE GRAAF. A mathematical introduction to Dirac's formalism. North-Holland Mathematical Library 36, 1986.

[EG2] A measure theoretical Sobolev lemma. J. Funct. Anal. 60 (1) 1985, pp. 95-101.

[EG3] A fundamental approach to the generalized eigenvalue problem for self-adjoint operators. J. Funct. Anal. 63 (1), 1985, pp. 74-85.

[EG4] Trajectory spaces, generalized functions and unbounded operators. Lect. Notes in Math. 1162, Springer-Verlag, Berlin etc., 1985.

[Fe] FEDERER H., Geometric measure theory. Die Grundlehren der mathematischem Wissenschaffen in Einzeldarstellungen, Band 153, Springer-Verlag, Berlin etc., 1969.

[Ro] ROBERTSON B., Introduction to field operators in quantum mechanics. Amer. J. Phys. 41, 1973, pp. 678-690.

[Sh] SVARTS, A.S., Mathematicheskie osnovy kvantovoj theorii polya. Moskva, Atomisdat, 1975, in russian.

[WZ] WHEEDEN, R.L., A. ZYGMUND, Measure and integral. Marcel Dekker Inc., New-York, 1974.

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INTEGRAL LATTICES,

in particular those of Witt and of Leech

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TABLE OF CONTENTS

1. Introduction

The Witt lattice W in $\mathbb{R}^8$, and the Leech lattice L in $\mathbb{R}^{24}$, are perhaps the most well-known of all integral lattices. They have significance for geometry, finite groups and combinatorics, whose multi-dimensional aspects in part are governed by W for dimensions $\leqslant 10$, and by L for dimensions $\leqslant 26$. In the present introductory survey we try to give the reader an idea about what is going on in these fields. In the appendix (Section 5) we point at the literature in other areas of application, such as the geometry of numbers, arithmetic groups, combinatorial geometry, algebraic geometry, representation of algebras, and the theory of superstrings in physics.

Let $\mathbb{R}^d$ denote a real vector space of dimension d, provided with a nondegenerate symmetric inner product $(\,,)$, which may be definite or indefinite. Call (x,x) the norm of $x \in \mathbb{R}^d$. A *lattice* in $\mathbb{R}^d$ is a free Abelian subgroup of rank d. A lattice is *integral* if the inner products of its vectors are integers (the coordinates of the vectors need not be integral). In indefinite space integral lattices are completely determined by their dimension, type and signature. The type is I or II according as there exists a lattice vector of odd norm or not. The classification of integral lattices in positive definite spaces is much more complicated, and essentially unknown. The classical notion of stereographic projection can relate the two cases. This is illustrated in Section 4. In the earlier Section 3 we concentrate on the vectors of norm 2 in the integral lattice. They are called roots since their perpendicular hyperplanes reflect the lattice. Section 2 gives a quick and concrete introduction to the lattices W and L, with the emphasis on the vectors of minimum norm: 240 of norm 2 for W, and $2\begin{bmatrix}28\\5\end{bmatrix}$ of norm 4 for L. We explain these numbers.

2. The lattices of Witt and of Leech

2.1. *Definitions*

Let $H=-I+S$ be a skew Hadmard matrix of size $4k$, that is, H has entries ± 1 and

$$HH^t = 4kl, \quad S+S^t = 0, \quad \mathrm{diag}(H-I) = -2I.$$

Following McKay [9], we consider the following matrix B of size $8k$, and we calculate B^tB:

$$B_{8k} := \frac{1}{\sqrt{k+1}}\begin{bmatrix}(k+1)I_{4k} & H_{4k}-I_{4k}\\ O_{4k} & I_{4k}\end{bmatrix}, \quad B^tB = \begin{bmatrix}(k+1)I & H-I\\ H^t-I & 4I\end{bmatrix}.$$

The integral linear combinations of the columns of B_{8k} constitute a *lattice* Λ_{8k} in $\mathbb{R}^{8k}$, which has the following properties:

Λ_{8k} is *integral* (all lattice vectors have integral pairwise inner products), since B^tB is an integral matrix;

Λ_{8k} is *unimodular* (the parallelepipedum of the basis vectors has volume one), since $\det B = 1$;

Λ_{8k} is *even* (the norms (x,x) of all lattice vectors are even), for odd k.

We shall restrict to the cases $k=1$ and $k=3$. For $k=1$ the lattice Λ_8 is the Witt lattice [14]. For $k=3$ the lattice is the Leech lattice [8].

2.2. The Witt lattices in $\mathbf{R}^8$

For $k=1$ we expose the matrix B, and some easy linear combinations of the 8 columns, which form the $240=8+8+16(4+4+6)$ vectors of minimum norm $(x,x)=\frac{1}{2}\times 4=2$ in the lattice.

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 2&0&0&0&-2&1&1&1\\ 0&2&0&0&-1&-2&1&-1\\ 0&0&2&0&-1&-1&-2&1\\ 0&0&0&2&-1&1&-1&-2\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}\begin{matrix} \pm2&0&0&\pm1&0\\ 0&0&\pm1&0&0\\ 0&0&\pm1&0&\pm1\\ 0&0&\pm1&0&\pm1\\ 0&\pm2&\pm1&0&0\\ 0&0&0&\pm1&0\\ 0&0&0&\pm1&\pm1\\ 0&0&0&\pm1&\pm1 \end{matrix}$$

These 240 vectors in $\mathbf{R}^8$ all have norm $(r,r)=2$ and inner products $(r,s)\in\{0,\pm1,\pm2\}$. They form the so-called *root system* E_8. The pairs of opposite vectors are on 120 lines in $\mathbf{R}^8$ which have angles 60° and 90°, so have cosines $\in\{\frac{1}{2},0\}$.

2.3. The Leech lattice in $\mathbf{R}^{24}$

For $k=3$, in dimension 24, we read from B^tB that the basis vectors all have norm 4. It is not difficult to show that all lattice vectors have norm at least 4. Furthermore, $(x\pm y,\, x\pm y)\geqslant 4$ implies

$$\forall_{x,y\in X}(x,y)\in\{0,\pm1,\pm2,\pm4\} \quad \text{for} \quad X:=\{x\in\Lambda\colon (x,x)=4\}.$$

We expose the vectors of X, the vectors of minimum norm 4. Apart from a common factor $1/\sqrt{8}$ they are as follows, cf. [4].

729×2^7 vectors of type $(\pm2)^8 0^{16}$, having an even number of minusses, and the 8 nonzero coordinates distributed over the 24 coordinates as the blocks of the Steiner 5 - (24,8,1) design; $\binom{24}{2}\times4$ vectors of type $(\pm4)^2 0^{22}$, all possibilities for the signs and for the 2 nonzero coordinates over 24 places;

24×2^{12} vectors of type $(\pm3)^1(\pm1)^{23}$, where the 3 runs through all 24 positions, and the minus signs follow the pattern of the extended Golay code.

Altogether these are $196560=2\binom{28}{5}$ vectors in X, and it will soon turn out that this is all. Since the vectors of X are pairwise antipodal and have norm 4, the antipodal pairs are on $\binom{28}{5}$ lines at angle ϕ with $\cos\phi\in\{0,\frac{1}{2},\frac{1}{4}\}$.

2.4. Inequalities for sets of lines

The following general theorem, which is of independent interest [6,7], explains why nice numbers such as $\binom{28}{5}$ and $\binom{10}{3}$ occur.

THEOREM 2.1. *In Euclidean space $\mathbb{R}^d$ there are at most $\binom{d+2s}{2s+1}$ lines having an angle ϕ selected from at most $s+1$ values, one of which being $\pi/2$.*

PROOF. Consider n lines through the origin in $\mathbb{R}^d$ whose angles ϕ have $\cos\phi \in \{0, \alpha_1, ..., \alpha_s\}$. Let X denote a set of n unit vectors, one along each of the lines. Then

$$(x,x) = 1 \quad \text{and} \quad (x,y) \in \{0, \pm\alpha_1, \pm\alpha_2, ..., \pm\alpha_s\},$$

for all $x,y \in X$. Now consider the n polynomials F_x, one for each $x \in X$, defined by

$$F_x(\xi) := (x,\xi)\prod_{i=1}^{s}\frac{(x,\xi)^2 - \alpha_i^2(\xi,\xi)}{1-\alpha_i^2}, \quad \xi \in \mathbb{R}^d.$$

These are independent polynomials. Indeed, suppose $\Sigma_{x \in X} C_x F_x = 0$, for $C_x \in \mathbb{R}$, then substitution of any $y \in X$ for the running variable ξ yields $F_x(y) = \delta_{x,y}$, hence $C_y = 0$. On the other hand, the polynomials F_x, $x \in X$ belong to the linear space $\mathrm{Hom}_k(\mathbb{R}^d)$ of the homogeneous polynomials in d variables of total degree $2s+1$. Hence their number cannot exceed

$$\dim \mathrm{Hom}_k(\mathbb{R}^d) = \binom{d-1+k}{d-1},$$

which proves the theorem. □

REMARK. As an illustration of the formula for the dimension, we demonstrate that there are $\binom{9}{2}$ monomials of degree 7 in 3 variables:

$$O \quad O \quad \varnothing \quad O \quad O \quad O \quad O \quad \varnothing \quad O \quad \text{indicates} \quad x_1^2 x_2^4 x_3^1.$$

REMARK. The number of lines at few angles is bounded by the present and similar inequalities, cf. [6]. For each of the following bounds the maximum is indeed achieved. We saw this already for the first two cases, and will encounter the other cases later

d :	24	8	23	7	23
α :	$0, \frac{1}{2}, \frac{1}{4}$	$0, \frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{3}$	$0, \frac{1}{3}$
$n \leqslant$	$\binom{28}{5}$	$\binom{10}{3}$	$\binom{24}{2}$	$\binom{8}{2}$	$\binom{25}{3}$
	98280	120	276	28	2300

3. Roots

3.1. Root systems

Let $\mathbb{R}^d$ denote real Euclidean space of dimension d. A *root system* in $\mathbb{R}^d$ is a set of vectors such that, for each pair of vectors,

$$(r,r) = 2 \quad \text{and} \quad (r,s) \in \{0,1,-1,2,-2\}.$$

Apart from the root system E_8, which was defined explicitly in 2.2, we have:

$$E_7 := \{x \in E_8 \colon (x,v)=0\}, \quad \text{for some } v \in E_8;$$

$$E_6 := \{x \in E_8 \colon (x \perp \text{star}\}, \quad \text{for some star;}$$

$$A_d := \{e_i - e_j \colon \{e_1, \ldots, e_{d+1}\} \text{ o.n. basis in } \mathbb{R}^{d+1}\};$$

$$D_d := \{\pm e_i \pm e_j,\ i \neq j \colon \{e_1, \ldots, e_d\} \text{ basis in } \mathbb{R}^d\}.$$

The numbers of nonzero vectors, and of lines connecting antipodal vectors, are as follows:

for	:	E_8	E_7	E_6	A_d	D_d
# vectors	:	240	126	72	$d(d+1)$	$2d(d-1)$
# lines	:	120	63	36	$\frac{1}{2}d(d+1)$	$d(d-1)$

THEOREM 3.1. *Following the definition above the only root systems are A_d, D_d, E_6, E_7, E_8 ($d \geq 1$).*

PROOF. See [2]. □

Root systems are important for the theory of integral lattices. Indeed, a reflection

$$x \mapsto x - 2\frac{(x,r)}{(r,r)}r, \quad x \in \mathbb{R}^d,$$

in the hyperplane orthogonal to a root r provides an automorphism $x' = x - (x,r)r$ of the lattice if $(r,r)=2$. Sometimes an integral lattice is generated by such reflections, but usually it is not. We will come back to this later.

3.2. Coxeter graphs, graphs with $\beta_{\min} = -2$

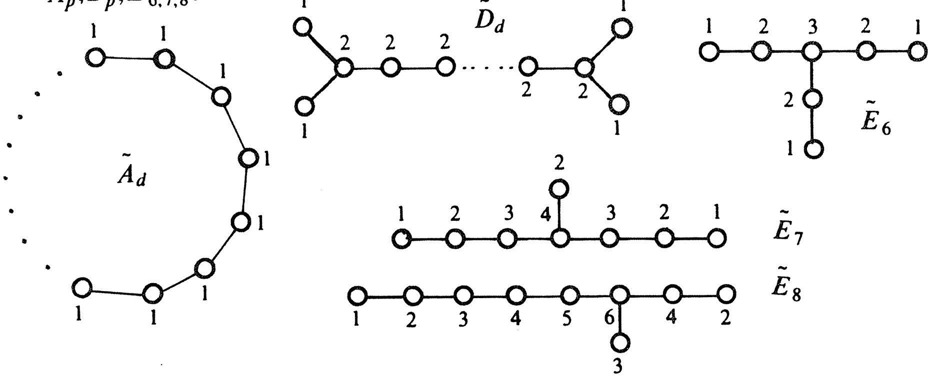
For any root system, let us consider the Gram matrix $2I + C$ of the inner products of the roots. This matrix has 2 on the diagonal, and entries $0, 1, -1, -2$ elsewhere. By permutation we produce submatrices $2I - A$ (off-diagonal entries 0 and -1 only) and $2I + B$ (off-diagonal entries 0 and 1 only).

$$2I+C = \left[\begin{array}{cccccccc} 2 & 0/-1 & & & & & & \\ & 2 & & & & & & \\ & & 2 & & & & & \\ & & & \ddots & & & & \\ & & & & 2 & & & \\ 0/1/-1 & & & & & 2 & & \\ & & & & & & 0/1 & \\ & & & & & & & 2 \end{array}\right] \quad \begin{array}{l} 2I-A, \ \alpha_{max}=2. \\ \\ \\ \\ \\ \\ 2I+B, \ \beta_{min}=-2. \end{array}$$

The Gram matrix $2I+C$ is positive semi-definite, and so are its principal submatrices $2I-A$ and $2I+B$. For the eigenvalues of the (1,0)-matrices A and B this implies: A has $\alpha_{max} \leqslant 2$ and B has $\beta_{min} \geqslant -2$. If $2I-A$ is singular then $\alpha_{max}=2$, and if $2I+B$ is singular then $\beta_{min}=-2$.

A *Coxeter graph* is a finite graph whose adjacency matrix A has maximum eigenvalue $\alpha_{max}=2$ (implicitly in [5]). The corresponding eigenvector x satisfies $Ax=2x$, hence $\Sigma_{j\sim i}x_j=2x_i$, for $i=1, ..., v$, that is, twice the value at the vertex i equals the sum of the values at the neighbouring vertices of i. Here are five examples of Coxeter graphs: two infinite families and three sporadic graphs. Each time the eigenvector is indicated, and one can check the property $Ax=2x$.

$\tilde{A}_p, \tilde{D}_p, \tilde{E}_{6,7,8}$:



The eigenvalue 2 is indeed the maximum eigenvalue since the eigenvectors are positive vectors (Theorem of Perron-Frobenius). The following theorem is not difficult to prove.

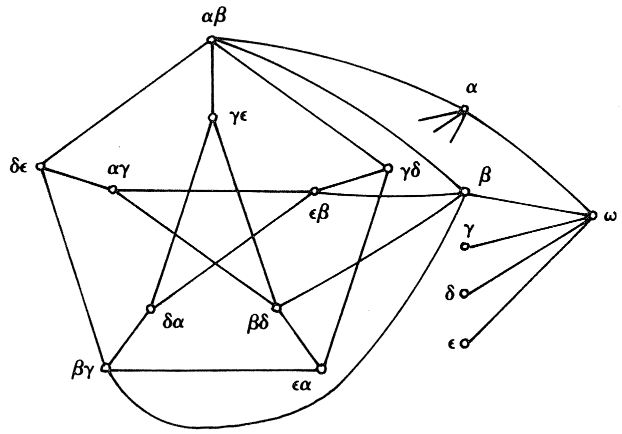
THEOREM 3.2. *The only connected Coxeter graphs are* $\tilde{A}_d$, $\tilde{D}_d$, $\tilde{E}_6$ $\tilde{E}_7$, $\tilde{E}_8$.

We turn to finite graphs whose adjacency matrix B has smallest eigenvalue -2. We refer to [3] for the following result:

THEOREM 3.3. *Graphs with $\beta_{\min}=-2$ have a matrix $2I+B$ which can be obtained from the root systems D_d or E_8.*

Graphs whose $2I+B$ is in the root system D_d, are (generalized) line graphs. For instance the graph (V,E) has a line graph which is represented by the set of vectors $\{e_i+e_j: \{i,j\}\in E\}$ in D_v.

Graphs whose $2I+B$ is in the root system E_8, are sporadic graphs. They include the graphs of Petersen on 10, Clebsch on 16, and Schläfli on 27 vertices. The vertices of the Petersen graph are represented by the 10 pairs out of 5 symbols, two vertices being adjacent iff the corresponding pairs are disjoint. The Clebsch graph has 6 further vertices, the 5 symbols and a vertex ω. The adjacencies are as indicated below. For a description of the Schläfli graph cf. [3].

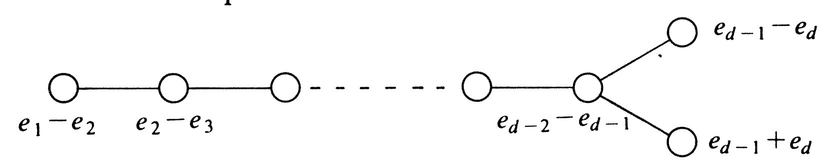


3.3. Root lattices

A *root lattice* is an integral lattice which is generated by norm 2 vectors. We given two examples, to be denoted by D_d and by E_8. Let $\{e_1,e_2, ...,e_d\}$ be an orthonormal basis in $\mathbb{R}^d$. Define

$$D_d := \left\{x\in\mathbb{R}^d: x_i\in\mathbb{Z},\ \sum_{i=1}^{d}x_i\in 2\mathbb{Z}\right\}.$$

This integral lattice is generated by the following d vectors of norm 2, whose Gram matrix is also exposed:



$$\begin{bmatrix} 2 & - & & & & & \\ - & 2 & - & & & & \\ & - & 2 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & 2 & - & - \\ & & & & - & 2 & 0 \\ & & & & - & 0 & 2 \end{bmatrix} = \text{Gram.}$$

The vectors are taken as the vertices of a graph, two vertices being adjacent (nonadjacent) if the vectors have inner product -1 (inner product 0). The resulting graph is a subgraph of the Coxeter graph $\tilde{D}_d$. The Gram matrix of the generating vectors has determinant 4, hence the lattice D_d is not unimodular but has det(basis) $=2$.

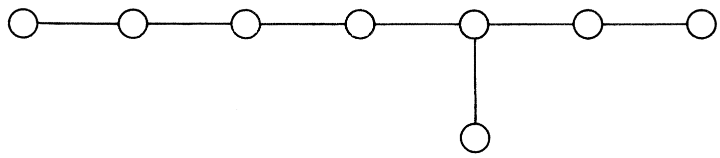
Our second example of a root lattice is

$$E_8 := \langle D_8, \tfrac{1}{2}(e_1+e_2+\cdots+e_8)\rangle_{\mathbf{Z}}.$$

Clearly $\frac{1}{2}(e_1+e_2+\cdots+e_8)$ has norm 2, and integral inner product with all vectors of the lattice D_8. Hence E_8 is integral and a root lattice. Since $2\mathbf{Z}_8 \supset D_8$ and $2E_8 \supset D_8$ we have

$$[\mathbf{Z}_8 : D_8]=2 \quad \text{and} \quad [E_8 : D_8]=2$$

for the indices, hence E_8 has det(basis) $=1$. In fact, E_8 is the integral unimodular even Witt lattice of Section 2.2. The graph of its basis reads



and is a subgraph of the Coxeter graph $\tilde{E}_8$.

Apart from D_d and E_8 there are further root lattices called A_d, E_7, E_6. The following classification of root lattices is related to Theorem 3.1 on root systems and 3.3 on graphs having smallest eigenvalue -2.

THEOREM 3.4. *Every root lattice is a direct sum of lattices A_d, D_d, E_d.*

This theorem has recently proved to be very useful in the classification of certain classes of graphs, cf. Appendix § 5.5.

3.4. Roots and the Leech lattice

There are no roots in the Leech lattice L since all vectors in L have norm $\geqslant 4$. However, roots are useful to describe deep holes in the Leech lattice. A *hole* in L is a ball whose boundary does, but whose interior does not contain vectors from L. It has been shown [4] that the Leech lattice contains *deep holes* of radius $\sqrt{2}$, and small holes of radius $<\sqrt{2}$.

Let h denote the centre of any deep hole, and let the boundary contain the distinct lattice points $0, z_2, z_3, ..., z_p \in L$. For $h_i := z_i - h$ we have:

$$4 \leqslant (z_i - z_j,\ z_i - z_j) = (h_i - h_j,\ h_i - h_j) = 2 + 2 - 2(h_i, h_j),$$

Hence (h_i, h_j) takes the values 0 and -1. The boundary points correspond to the points $h_1, h_2, ..., h_p$ having Gram matrix $[(h_1, h_j)] = 2I - A$, where A is a (1,0) matrix. Therefore, the boundary points of a deep hole correspond to a Coxeter graph. From Theorem 3.2 we know that this graph is a disjoint union of graphs of type $\tilde{A}_d$, $\tilde{D}_d$, $\tilde{E}_d$. Conway and Sloane [4] found that there are 23 types of deep holes, such as

$$\tilde{A}_1^{24},\ \tilde{A}_2^{12},\ \tilde{A}_3^{8},\ ..., \tilde{D}_{16}\tilde{E}_8, \tilde{D}_{24},$$

and that the Leech lattice may be reconstructed from each deep hole. Borcherds [1] investigated the 284 types of small holes. Neumaier [10] looked at the relations with combinatorial objects.

3.5. Line systems in the Leech lattices

The Leech lattice has an automorphism group of the order

$$8\ 315\ 553\ 613\ 086\ 820\ 000\,.$$

The quotient over $\{\pm 1\}$ is Conway's simple group Con. 1, cf. [4]. The Leech lattice contains several substructures whose automorphisms form simple groups as well. We mention some of these.

Given $e \in L$ with $(e,e)=4$, we ask for all $z \in L$ such that $(z,z) = 4 = (z-e,\ z-e)$. These vectors are in the flat $(z,e)=2$ of dimension 23, and $(z - \frac{1}{2}e,\ z - \frac{1}{2}e) = 3$. The inner product of two such $z, z' \in L$ can take the following values:

$$\begin{array}{lcl} (z,z') & = & 4\ \ 2\ \ 1\ \ \ 0\ \ -1\ \ -2\ \ -4\,, \\ (z - \frac{1}{2}e, z' - \frac{1}{2}e) & = & 3\ \ 1\ \ 0\ \ -1\ \ -2\ \ -3\ \ -5\,. \end{array}$$

However, $(z,z') \neq -4$ since $(z - \frac{1}{2}, z' - \frac{1}{2}e) = -5$ conflicts with Cauchy-Schwarz, and $(z,z') \neq -1$ since $(z + z' - e,\ z + z' - e) \neq 2$. There remains $(z - \frac{1}{2}e, z' - \frac{1}{2}e) \in \{\pm 3, \pm 1, 0\}$. Hence the lines spanned by $z - \frac{1}{2}e$ are in 23-space and have angles ϕ whose cosine equals 0 or 1/3. Following Theorem 2.1 there are at most 2300 of such lines. It turns out that this maximum is achieved by the present set. The automorphism group of this set of lines is Con. 2.

Given $f \in L$ with $(f,f)=6$, we ask for all $z \in L$ such that $(z,z)=4=(z-f,\, z-f)$. These vectors are in the flat $(z,f)=3$ of dimension 23, and $(z-\frac{1}{2}f, z-\frac{1}{2}f)=5$. As above for two such $z,z' \in L$ there remains $(z-\frac{1}{2}f, z'-\frac{1}{2}f) \in \{\pm 5, \pm 1\}$. Hence the lines spanned by $z-\frac{1}{2}f$ are in 23-space and have one angle ϕ with $\cos\phi = 1/5$. Again, it turns out that the present construction yields the maximum number 276 of such lines according to Theorem 2.1. The automorphism group of this set of lines is Con. 3. It acts 2-transitively on the lines.

There are other simple groups involved in the Leech lattice. The simple group of McLaughlin is the automorphism group of the graph on 275 vertices which is obtained as follows. Take the vector e, with $(e,e)=5$, along any one of the 276 equiangular lines at $\cos\phi = 1/5$ in 23-space as mentioned above. This vector determines 275 vectors z on the remaining lines with $(z,z)=5$, $(z,e)=1$. These vectors z have inner products ± 1, and determine McLaughlin's graph.

Another example is the simple group of Higman and Sims, which is the automorphism group of 176 equiangular lines at $\cos\phi = 1/5$ in 22-space, again a system of lines involved in the Leech lattice.

4. Integral lattices in indefinite space

4.1. How many lattices?

Integral, unimodular, even lattices in Euclidean d-space only exist for $d \equiv 0 (\mathrm{mod}\, 8)$. But the number of such nonisomorphic lattices explodes already for $d \geqslant 32$:

d	:	0	8	16	24	32
$\#$ in $\mathbb{R}^d$	:	–	1	2	24	millions

This is sharp contract with the situation in indefinite space $\mathbb{R}^{d+1,1}$, where integral unimodular even lattices exist and are *unique*, for $d \equiv 0 (\mathrm{mod}\, 8)$ only, cf. [12]. Ch. V,

$\#$ in $\mathbb{R}^{d+1,1}$	:	1	1	1	1	1
lattices	:	H	$H+W$	$H+W+W$	$H+W+W+W$	$H+4W$

Here W denotes the Witt lattice E_8 in Euclidean 8-space. H denotes the integral unimodular even lattice in the hyperbolic plane $\mathbb{R}^{1,1}$ which is defined as follows.

In the plane take a symplectic basis $\{e_1, e_2\}$ and define inner products by linear extension of

$$(e_1,e_1) = 0 = (e_2,e_2), \quad (e_1,e_2) = 1 = (e_2,e_1);$$

$$(x_1e_1+x_2e_2, y_1e_1+y_2e_2) = x_1y_2+x_2y_1, \ \|x_1e_1+x_2e_2\|^2 = 2x_1x_2.$$

The unit circle in $\mathbf{R}^{1,1}$ has the equation $2|x_1x_2|=1$, and the light cone consists of the two coordinate axes. The lattice $H=\mathbf{Z}^{1,1}$ consists of the vectors with integer coordinates, provided with the inner product defined above.

Theorem 4.1. *In* $\mathbf{R}^{d+1,1}$ *the only integral unimodular lattice are* $\mathbf{Z}^{d+1,1}$ *and* $H+\frac{1}{8}dW$ *(even lattices, for* $d\equiv 0(\mathrm{mod}\ 8)$*).*

Proof. Cf. Theorem 4, 5, 6, Chapter V of [12].

Warning. The reader should realize that the isomorphism $H+L_1\cong H+L_2$ by no means implies an isomorphism of the lattices L_1 and L_2.

4.2. Lorentz space $\mathbf{R}^{p,1}$

Real indefinite space $\mathbf{R}^{p,1}$ is $(p+1)$-space $\mathbf{R}^{p+1}$ provided with the inner product

$$(x,y) = -x_0y_0+x_1y_1+\cdots+x_py_p, \quad \text{for} \quad x=(x_0;\, x_1,\cdots,x_p).$$

The vectors in $\mathbf{R}^{p+1}$ with $(x,x)=0$ constitute the cone

$$C\colon x_1^2+\cdots+x_p^2 = x_0^2.$$

The line $<a>_{\mathbf{R}}$ spanned by the vectors a is inside, on, outside the cone, according as (a,a) is negative, zero, positive, respectively. Then the hyperplane $a^{\perp}$ is Euclidean, degenerate, Lorentzian, respectively. The plane $<a,b>_{\mathbf{R}}$ spanned by $<a>\neq<b>$ is passing, tangent, intersecting the cone, according as $(a,b)^2-(a,a)(b,b)$ is negative, zero, positive, respectively. Hence if $<a,b>$ is passing then we may interpret:

$$\frac{|(a,b)|}{\sqrt{(a,a)(b,b)}} = \cos\phi, \quad \phi = \mathrm{angle}(<a>,<b>),$$

the angle between the lines $<a>_{\mathbf{R}}$ and $<b>_{\mathbf{R}}$ outside the cone. On the other hand, if $<a,b>$ is intersecting and $<a>$ and $<b>$ are inside the same nappe of C, then we may interpret:

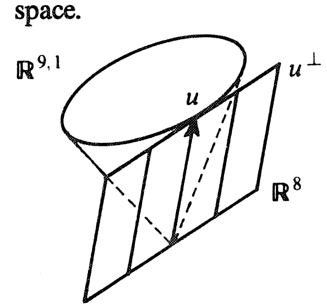
$$\frac{(a,b)}{\sqrt{(a,a)(b,b)}} = \cosh d, \quad d = \mathrm{dist}(<\ >,<\ >),$$

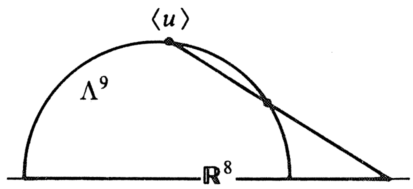
the distance between $<a>_{\mathbf{R}}$ and $<b>_{\mathbf{R}}$.

The last case leads to Bolyai-Lobatchevski geometry Λ^p. The hyperbolic points are the lines inside C, the ideal points are the lines on C, the hyperbolic lines are the sections of the planes inside C, etc., and the distance between points is as defined above. However, rather than pursuing hyperbolic geometry Λ^p, we shall be interested in the geometry outside of the cone, cf. [11].

Example. Take $p \geqslant 9$.The $\begin{bmatrix} p+1 \\ 3 \end{bmatrix}$ vectors of type $(1;\ 1^3 0^{p-2})$ and the $\begin{bmatrix} p+1 \\ 2 \end{bmatrix}$ vectors of type $(0;\ 1,-1,0^{p-1})$ all have inner product 2 with themselves, and inner products 1, 0, -1 mutually. Hence these $\begin{bmatrix} p+2 \\ 3 \end{bmatrix}$ vectors and their negatives form a root system in $\mathbb{R}^{p,1}$. Indeed, all vectors are situated in the hyperplane perpendicular to the vector $w := (3;\ 1^{p+1})$, which is outside the cone since $p \geqslant 9$. So we have $\begin{bmatrix} p+2 \\ 3 \end{bmatrix}$ lines in $\mathbb{R}^{p,1}$ at 60° and 90°.

Example. The case $p=8$ is special, since now w is on the cone. The hyperplane $w^{\perp}$ is tangent to the cone and contains w as an isotropic vector. The quotient $w^{\perp}/<w>$ yields Euclidean 8-space, and the 120 lines at 60° and 90° yield a root system E_8. The present construction amounts to *stereographic projection*, and can be pictured in the vector space $\mathbb{R}^{9,1}$ as well as in projective 9-space.

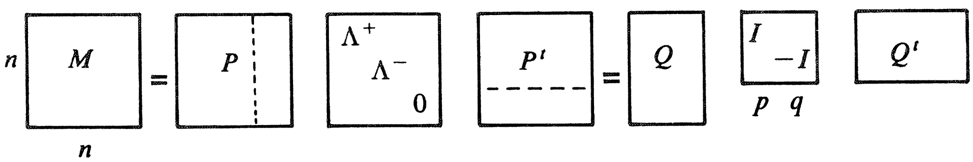




The introduction of indefinite metric is quite natural, and will bring about relations to graph theory. More general we have [11]:

Theorem 4.2. *Any $n \times n$ real symmetric matrix may be viewed as the Gram matrix of n vectors in indefinite space $\mathbb{R}^{p,q}$.*

Proof. Any symmetric matrix M is congruent to a diagonal matrix, and the diagonal entries are the eigenvalues of M:



The diagonal matrix Λ_p^+ of the positive eigenvalues can be made into I_p, and the diagonal matrix Λ_q^- can be made $-I_q$, by distributing $\sqrt{\lambda}$ (resp. $\sqrt{-\lambda}$) over P and P^t. Deleting irrelevant parts we obtain the $n \times (p+q)$ matrix Q and its transposed. Now we can read off that the n row vectors of Q have indefinite inner products equal to the entries of M.

COROLLARY 4.3. *Let λ_2 denote the second largest eigenvalue of the adjacency matrix A of a graph on n vertices. Then $\lambda_2 I - A$ is the Gram matrix of n vectors in $\mathbb{R}^{p,1}$.*

EXAMPLE 4.4. Let N denote the 10×15 incidence matrix of the vertices and the edges of the Petersen graph, cf. § 3.2. The eigenvalues of the 25×25 matrix

$$G := \begin{bmatrix} 2I & -N \\ -N^t & 2I \end{bmatrix} = 2I - \begin{bmatrix} 0 & N \\ N^t & 0 \end{bmatrix}$$

are related to the eigenvalues of the 10×10 matrix

$$NN^t = 2I + A.$$

Since the adjacency matrix A of the Petersen graph has the eigenvalues 3^1, 1^5, $(-2)^4$, the matrix G has the eigenvalues

$$(2+\sqrt{6})^1,\ 4^5,\ 3^4,\ 2^5,\ 1^4,\ 0^5,\ (2-\sqrt{6})^1,$$

so 19 positive, 5 zero and one negative eigenvalue. Theorem 4.2 implies that G is the Gram matrix of 25 vectors in $\mathbb{R}^{19,1}$.

REMARK 4.5. Example 4.4 has interesting consequences, cf. [13], [11]. The 25 vectors in $\mathbb{R}^{19,1}$ (which can be given integral coordinates) have norm 2, mutual inner products 0 and -1, and span the space. They form the largest set in $\mathbb{R}^{19,1}$ relative to this property. The 25 vectors generate an integral lattice which is even, but not unimodular since $19-1 \not\equiv 0 \pmod 8$. We can make it unimodular by sticking in a vector of norm 1 which has inner products 0 and -1 with the 25 vectors. The new lattice is integral and unimodular. Both lattices in $\mathbb{R}^{19,1}$ have an automorphism group which is generated by reflections. In other words, both lattices are *reflexive*, that is, their automorphism group contains a subgroup of finite index which is generated by reflections, and the roots span the space. For higher dimensions $\mathbb{R}^{p,1}$, $p > 19$, integral lattices need no longer be reflexive [13].

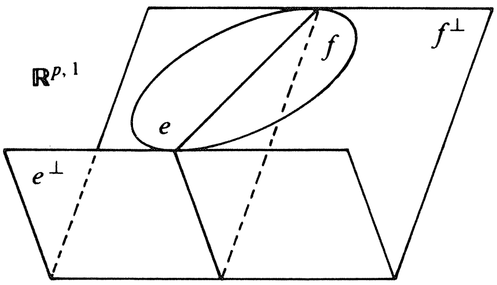
4.3. Integral lattices in $\mathbb{R}^{p,1}$

By Theorem 4.1 there are unique integral unimodular lattices in $\mathbb{R}^{p,1}$. These are:

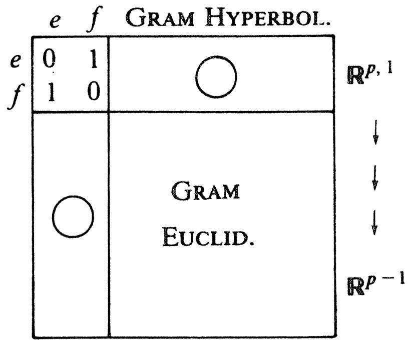
$$I_{p,1} := \mathbb{Z}^{p,1}; \quad II_{p,1} := \langle x \in \mathbb{Z}^{p,1} : \Sigma x_i \in 2\mathbb{Z},\ \tfrac{1}{2}(1 : 1^p) \rangle_{\mathbb{Z}}$$

and the even integral unimodular lattice $II_{p,1}$ only exists for $p \equiv 1 \pmod 8$. Stereographic projection relates the unique lattice $I_{p,1}$ to the odd integral unimodular lattices in Euclidean $\mathbb{R}^{p-1}$, and the unique $II_{p,1}$ to the even integral unimodular lattices in $\mathbb{R}^{p-1}$. We explain this for the even case [13]. Then $\mathbb{R}^{p,1}$ contains a hyperbolic plane $H = \langle e, f \rangle_{\mathbb{R}}$. Hence in the Gram matrix of any basis through e and f in $\mathbb{R}^{p,1}$ we may split off a submatrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; perpendicular to H we are left with a positive definite Gram matrix for the remaining

Euclidean $(p-1)$-space. This is illustrated as follows, both by a geometric picture and by a dissected matrix:



$e^{\perp} \cap f^{\perp} = e^{\perp}/\langle e \rangle = Eucl_{p-1}$



Now it is easy to read off the following common properties of the lattice $I_{p,1}$ (take $\epsilon=1$) or $II_{p,1}$ (take $\epsilon=0$) on the one hand, and the Euclidean lattices on the other hand:

$I_{p,1}$ or $II_{p,1}$	unique	odd	even	unimod.	reflex.
$\begin{bmatrix} 0 & 1 \\ 1 & \epsilon \end{bmatrix} + \mathrm{Eucl}_{p-1}$	all	odd	even	unimod.	reflex.

As an illustration, all integral unimodular even lattices in Euclidean space

E_8 in $\mathbb{R}^8$; E_8+E_8 and E_{16} in $\mathbb{R}^{16}$; 23 Niemeier and Leech in $\mathbb{R}^{24}$

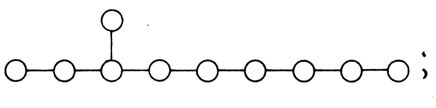
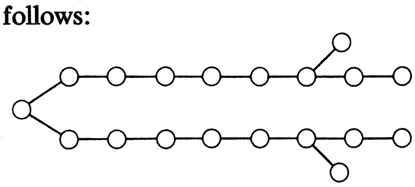
are obtained from the unique

$II_{9,1}$; $II_{17,1}$; $II_{25,1}$, respectively.

In order to obtain the various Euclidean lattices one has to select the right north pole e for the stereographic projection, for instance:

$e=(3;1^9)$; $e=(3;1^9 0^8)$; $e=(5;3\,1^{16})$; $e=(5;1^{25})$ etc. $e=(70;0,1,2,\cdots,24)$ for the Leech lattice.

We refer to [4], [13] for these coordinates and for a description of the automorphism groups of the indefinite lattices. Finally, we mention [4] that the Coxeter graph for $II_{9,1}$, $II_{17,1}$, $II_{25,1}$, reads as follows:

 ; 

; the Leech lattice.

5. APPENDIX

On further applications of integral lattices

We given a sample of areas where integral lattices have been applied. There will be no details, no completeness, and only a selected set of references.

5.1. Coxeter-Dynkin diagrams

The ubiquity of Coxeter-Dynkin diagrams (an introduction to the A-D-E problem), by M. HAZEWINKEL, W. HESSELINK, D. SIERSMA, F.D. VELDKAMP, Nieuw Archief Wiskunde *25* (1977), 257-307.

5.2. Geometry of numbers

This is the theory of the behaviour of geometric bodies with respect to lattices. The theory was created by Minkowski, and recently surveyed in:

P.M. GRUBER, C.G. LEKKERKERKER, Geometry of numbers, second edition, North-Holland (1987).

5.3. Arithmetic groups

Further results on reflexive integral lattices, cf. 4.2, are:

- Unimodular $(p, 1)$-lattices are reflexive iff $p \leqslant 19$; unimodular Euclidean $(p-1)$-lattices are reflexive iff $p-1 \leqslant 18$.
- There are no reflexive $(p, 1)$-lattices for $p \geqslant 30$.
- All (p,q)-lattices are reflexive for $p \geqslant 2$ and $q \geqslant 2$.

E.B. VINBERG, Discrete reflection groups in Lobachevsky spaces, Proc. I.C.M. Warszawa (1983), 593-601.

V.V. NIKULIN, Reflection groups in Lobachevsky spaces and algebraic surfaces, Proc. I.C.M. Berkeley (1986), Abstract p. 130.

5.4. Algebraic geometry

Integral lattices play a role in the following subjects:

Homology of manifolds, cf.

J. MILNOR, D. HUSEMOLLER, Symmetric bilinear forms, Springer (1973).

Singularities of hypersurfaces, cf.

E. BRIESKORN, Milnor lattices and Dynkin diagrams, Proc. Symp. Pure Math. A.M.S. *40* (1983), 153-165.

K_3-surfaces, cf.

I. DOLGACHEV, Integral quadratic forms, applications to algebraic geometry (after V. Nikulin), Seminaire Bourbaki 35^e^ annĕe No. 611 (1983), 251-278.

5.5. Combinatorial theory

Recent characterizations by use of root lattices, and a future general reference:

A. NEUMAIER, Characterization of a class of distance regular graphs, J. Reine Angew. Math. *357* (1983), 182-192.

P. TERWILLIGER, Root systems and the Johnson and Hamming graphs, Europ. J. Combin. *8* (1987), 73-102.

A.E. BROUWER, A. COHEN, A. NEUMAIER, Distance regular graphs, Springer (1989).

5.6. Representations of algebras
C.M. RINGEL, Tame algebras and integral quadratic forms, Lecture notes in mathematics 1099, Springer (1984).

5.7. Superstrings
Integral lattices and spaces of dimensions 10 and 26 play a crucial role in the theory of superstrings:
M.B. GREEN, J.H. SCHWARZ, E. WITTEN, Superstring theory, two volumes, Cambridge Univ. Press (1987).
P. GODDARD, D. OLIVE, Algebras, lattices and strings, pp. 51-96 in Vertex operators in mathematics and physics, ed. J. Lepowsky, S. Mandelstam, I.M. Singer, Math. Sci. Res. Inst. Publ. Vol. 3, Springer (1985).
J. THIERRY-MIEG, Anomaly cancellation and Fermionisation in 10-, 18- and 26-dimensional superstrings, Physics Letters B, *171* (1986), 163-169.

REFERENCES

1. R. BORCHERDS (1985). The Leech lattice. Proc. Roy. Soc. London A *398*, 365-376.
2. N. BOURBAKI (1968). Groupes et algèbres de Lie, Chap. 4,5,6, Hermann.
3. P.J. CAMERON, J.M. GOETHALS, J.J. SEIDEL, E.E. SHULT (1976). Line graphs, root systems and elliptic geometry, J. of Algebra *43*, 305-327.
4. J.H. CONWAY, N.J.A. SLOANE (1988). Sphere packings, lattices and groups, Springer.
5. H.S.M. COXETER (1973). Regular polytopes, third edition, Dover.
6. P. DELSARTE, J.M. GOETHALS, J.J. SEIDEL (1975). Bounds for systems of lines, and Jacobi polynomials, Philips Research Reports *30*, 91-105.
7. T.H. KOORNWINDER (1976). A note on the absolute bound for systems of lines, Proc. Kon. Nederl. Akad. Wetensch. A *79* (= Indag. Math. *38*), 152-153.
8. J. LEECH (1967). Notes on sphere packings, Canad. J. Math. *19*, 251-267.
9. J. MCKAY (1973). A setting for the Leech lattice, pp. 117, 118 in Finite groups 72, ed. T. Gagen, M.P. Hale, E.E. Shult, North-Holland.
10. A. NEUMAIER (1987). Some relations between roots, holes, and pillars, Europ. J. Combin. *8*, 29-33.
11. A. NEUMAIER, J.J. SEIDEL (1983). Discrete hyperbolic geometry, Combinatorica *3*, 219-237.
12. J.P. SERRE (1970). Cours d'arithmétique, Presses univers. France.
13. E.B. VINBERG (1972). On unimodular integral quadratic forms, Funct. Anal. Appl. *6*, 105-111 (English trl.).
14. E. WITT (1941). Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe, Abhandl. Math. Sem. Hamburg *14*, 289-322.

Mathematical aspects of supersymmetry and anticommuting variables

I. The one-dimensional Wess-Zumino model

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Table of Contents

Abstract

In this first paper we give a rigorous account of the one-dimensional Wess-Zumino model, in order to establish basic mathematical concepts and to elucidate on an elementary level such matters as the connection between infinitesimal and non-infinitesimal supersymmetry transformations, the rôle of auxiliary fields and the superspace/superfield formalism and its group theoretical origin. This will be useful for the next paper here we shall study the mathematical structure of supersymmetric field theory in general.

I. Introduction

A. Supersymmetry and anticommuting variables

In the mid-seventies the possibilities of a new type of symmetry was discovered in particle physics. It was called *supersymmetry* and as an extension of standard Poincaré symmetry its main physical feature was the remarkable fact that it connected bosons and fermions. It attracted immediately wide-spread attention and its theoretical appeal has remainded considerable. Lack of experimental confirmation so far has not impeded its development. In all the main speculative ideas in particle physics in the last few years, including string theory, some sort of supersymmetry has been an important ingredient.

A characteristic mathematical aspect of super symmetry field theory, and at the same time the means by which it has been able to escape the restrictions of no-go theorems such as that of Coleman and Mandula, is the presence of so-called *anticommuting variables* (anticommuting *c*-numbers, Grassmann variables). In various places of the theory, in particular already at the classical level, one employs instead of ordinary real or complex numbers unspecified anticommuting numberlike objects. The general outcome of this is a far-reaching formal unification of boson and fermion aspects which makes in particular the formulation of the new boson-fermion symmetry possible.

As mathematics much of this is still at a heuristic stage. It is not hard to give a mathematical definition of anticommuting numbers as an isolated concept. An obvious possibility is to see them as odd elements of a Grassmann algebra, a rigorous and elementary mathematical notion. The anticommuting numbers generate as key objects, in an explicit or implicit manner, a general formalism. In this various new ideas arise quite naturally, such as superspace and supergroups. One obtains in fact a full "anticommuting" version of analysis and differential geometry. It is therefore of greater interest to understand more completely the mathematics of this total formalism, together with the way it is applied in physical theories.

In general high-energy physicists have not been worried excessively by the heuristic nature of all this. Nevertheless the situation is unsatisfactory from a more fundamental point of view. The matter is also of independent mathematical interest. There is an underlying complex of new mathematical ideas that deserves to be made explicit and rigorous.

B. The situation in the literature and the rôle of explicit examples

The literature on the subject falls into two different and widely separated categories. On one side there is a very extensive and still rapidly growing physical literature on super symmetry, supergravity, etc. in which anticommuting variables are used in an imaginative, often intrigueing but still in the main heuristic way. On the other side one finds a much smaller body of mathematically oriented work in which general foundations are studied rigorously. Between the two there is a striking difference in language and consequently only a very limited interaction. The connection between the explicit theories from the mainstream physical literature and the ideas from the mathematical work, some of which is quite advanced, remains unclear, in particular at the basic conceptual level. Clarification of the situation in this respect will be one of the main purposes of this and subsequent papers.

In the physical literature a great number of models of supersymmetric field theory of varying complexity have been studied in detail. There is at present a large gap between this wealth of explicit heuristic material and the mathematical work in which not only things are phrased very differently but which is also on a much more general level.

In this paper we start a program of studying standard examples of supersymmetric theories in an explicit and mathematically rigorous way. As we shall discuss more extensively in section II, there are basically two mathematical approaches to the formalism of anticommuting variables. There is one approach, sometimes called *"geometric"*, and for which we also shall use the term *"extended"*, in which anticommuting numbers are used explicitly as well-defined objects. Starting from physics it seems to be the most obvious way of giving a precise meaning to the heuristic formulas there. In the other approach the algebraic operations to which anticommuting numbers give rise are more important than the anticommuting numbers themselves. It is formulated in terms of $\mathbb{Z}_2$-graded algebras, and is therefore called *"algebraic"*. For reasons that will become clear later the term *"minimal"* will also be used by us. It seems to us that this formalism contains the essential mathematical ideas of supersymmetry and anticommuting variables in a more economical manner. The connection with the physics literature is however less immediate and not much explicit physical theory has been translated in this language yet.

Our plan in this paper is to discuss first the mathematical aspects of a simple model from supersymmetric field theory in the more obvious extended formalism. In the next paper important elements of a "minimal" version of the model will be extracted from the results. This will then lead to a transparent algebraic structure which will serve as a skeleton of more general supersymmetric field theories.

C. The Wess-Zumino model in one-dimensional space-time

There is a model which stands at the beginning of the subject of supersymmetry and which exhibits most of the features that one has learned to recognize as typical for supersymmetric theories. All further developments are based on it: Extended supersymmetry, supersymmetric gauge theories, super gravity, etc. This is the model introduced in 1974 by Wess and Zumino [1]. See for this and for a general introduction to supersymmetry Refs. [2], [3, 4] and more recently Ref. 5. As a physical theory the Wess-Zumino model describes a system of three different types of neutral particles, two with spin 0, and one with spin ½, interacting with each other, and of equal mass. It contains a scalar and a pseudo-scalar field for the bosons, a Majorana spinor field for the fermions and some auxiliary fields. The Lagrangian is such that the action is not only Poincaré invariant in the usual way, but also invariant under a larger group. Formulated infinitesimally this means invariance with respect to an additional set of variations which mix the boson and fermion fields.

This is supersymmetry. To make this work the variations have to contain formal anticommuting parameters and the fermion fields, as classical fields, must be regarded, again formally, as anticommuting objects.

Most of the standard texts on supersymmetric give the impression that in order to grasp the basic ideas of supersymmetry it is necessary first to go through a considerable amount of technicalities connected with Poincaré symmetry. This seems to us mistaken. In the first place and in its barest mathematical essence supersymmetry can be seen as an extension of translation symmetry. Of course in a relativistic theory these "supertranslations" have to be combined properly with the homogeneous Lorentz transformations. This is not completely trivial and requires some attention just as for ordinary space-time translations. It is at this stage, which we consider conceptually as secondary, that one needs the formulas for the generators of the Poincaré Lie algebra, definitions of Majorana and Weyl spinors, etc. etc. For this reason we think that it is of some use the study first a simpler one-dimensional version of the Wess-Zumino model, as a prelude to the discussion of the standard version in four-(or possibly higher) dimensional space-time. In the one-dimensional model the fields depend only on the time variable t. The number of fields is reduced: A boson field, a two component spinor field, and in what might be called the "off-shell" version, a single auxiliary field. All formulas and expressions become of course much simpler. The main advantage however is the fact that the Poincaré group as fundamental symmetry group has been reduced to the one-dimensional group of time translations. This simplification, drastic as it may seem, leaves enough of the basic mathematical ideas of supersymmetry as an extension of ordinary symmetry and in particular of such things as the rôle of auxiliary fields and of the possibilities of a superspace-superfield formulation.

The one-dimensional Wess-Zumino model has been discussed as a model for spontaneous symmetry breaking in supersymmetric quantum mechanics by SALOMONSON and VAN HOLTEN [6], following a suggestion of WITTEN [7]. Our work on this model has a different purpose and the results are therefore complementary to theirs and moreover of a different character.

D. Classical and quantum theory

In the formalism of field theory such as it is used in particle physics one should maintain a clear conceptual distinction between the classical and the quantum level. Classical field theory is mathematically speaking a theory of real or complex valued functions on space-time, extended for the purpose of supersymmetric to encompass the case where ordinary numbers are replaced by anticommuting variables in a manner to be made precise. It is not necessary for the classical field to have a direct physical interpretation. In our discussions here such an interpretation is in fact irrelevant. It is the quantum field that gives the proper physical description of particles and their interaction. For its construction we need the classical fields as a mathematical tool.

Setting up a particular field theory means in the first place choosing a set of fields together with transformation properties under certain symmetry groups and secondly finding a Lagrangian in terms of these fields which leads to an invariant action. Although all this has important consequences at the final quantum level of the theory, it is nevertheless essentially classical. This is in particular very obviously so when one employs the path integral quantization procedure.

Supersymmetric field theory is no exception to this general situation. Some of the typical aspects of supersymmetry are classical, in the sense given above, and should therefore be discussed as such, other aspects are true quantum aspects. Most of the physics literature does not care to make this distinction, which adds to the difficulty of making the mathematics precise. The mathematical literature, except that on graded Lie algebras, so far is only relevant to the classical level, a fact which is however seldom mentioned.

Four-dimensional quantum field theory with non-trivial interaction is still fraught with fundamental mathematical problems. In fact it can be said that as a mathematical theory it does not yet exist. In a rigorous discussion of quantum aspects of supersymmetry it may have some advantages not to get immediately entangled in these more conventional problems and this is an additional inducement to look first at supersymmetric field theory in one dimension. As quantum theory it is essentially quantum mechanics for which a fully satisfactory Hilbert space formulation is available.

We intend to keep a clear distinction between classical and quantum aspects. In general different problems should be discussed separately and conceptually one should proceed from the simplest situation to the more complicated ones. We shall therefore in this paper begin the discussion of the mathematical structure of supersymmetric field theory on the rather sober level of the one-dimensional classical Wess-Zumino model.

E. Contents of the paper

The contents of the rest of this paper can be summarized as follows: In section II we comment in somewhat greater detail on the two distinct mathematical approaches to anticommuting variables mentioned in part B of this section. In section IIII we introduce the one-dimensional Wess-Zumino model and give formulas and properties in a language close to that of ordinary classical field theory, followed by remarks on mathematical aspects. In section IV we show how supersymmetry, infinitesimally formulated in terms of variations, can be regarded as a group symmetry. We give the group action on the fields and also the supersymmetry group itself as an abstract group. The rôle of the introduction of an auxiliary field in this respect is made clear. On the basis of the supergroup found in section IV, we rewrite in section V the model in the language of superspace and superfields. In the concluding remarks in section VI we indicate how the result of section V will be used as a starting point for a more intrinsic algebraic version of the model in a second paper. Those elements of the Berezin calculus that that we need in this paper will be collected in an appendix.

II. Two approaches to anticommuting variables

As we have already indicated one finds in the mathematically oriented literature two essentially different ways of looking at the heuristic phenomenon of anticommuting numbers. These can be characterized in the following general way: In the *extended* or *geometric* approach one tries to remain as close as possible to the heuristic formulas from physics. This means in particular that one uses anticommuting numbers explicitly as mathematically well-defined objects, namely as the odd elements of a certain unspecified but fixed Grassmann algebra. In the *minimal* or *algebraic* approach the physical formulas are taken less literally. Anticommuting numbers are seen as a heuristic bookkeeping device for keeping track of manipulations in an underlying algebraic structure. It is this structure that is given a rigorous meaning. The anticommuting numbers themselves are absent from this formulation.

One usually explains the two approaches in the context of differential geometry of superspace. See e.g. Refs. [8] and [9]. In the geometric approach one has *supermanifolds*, first rigorously defined by Rogers [10, 11]. An (m,n)-dimensional supermanifold is roughly a manifold covered by patches of local coordinates, not in terms of real or complex numbers, but of m-tuples of (commuting) even and n-tuples of (anticommuting) odd elements from the fixed Grassmann algebra $\mathfrak{B}$ that is typical for the extended picture. There is by now an extensive mathematical literature on further developments and variations on this basic idea. Characteristic papers in this respect are those by Rogers [10, 12, 13], Jadczyk and Pilch [14], Hoyos, Quiros, Ramirez-Mittelbrunn and De Urries [15], Boyer and Gitler [16], Picken and Sundermeyer [17], Rabin and Crane [18] and Rothstein [19]. An imaginative although not fully rigorous account of the subject can be found in the book of De Witt [20]. In the algebraic approach to anticommutive differential geometry the main concept is that of a *graded manifold* as defined and further developed by Berezin and Leites [21, 22], Kostant [23] and Batchelor [24, 8, 9, 25]. A (m,n)-dimensional graded manifold is an m-dimensional ordinary, i.e. real C^∞ manifold on which the structure sheaf, i.e. the sheaf of commutative algebras of local C^∞ functions has been generalized to a sheaf of graded-commutative algebras. Locally these algebras are isomorphic to the tensor product of the algebra of C^∞ functions on an open set and a Grassmann algebra which is generated by n generators. It should be stressed that this Grassmann algebra has nothing to do with the unspecified Grassmann algebra, which typically appears in the extended/geometric picture.

The essential difference between the two approaches can already be understood on a level that is more elementary than that of differential geometry and manifolds. This is the level of what might be called *analysis* and *linear algebra* of *anticommuting variables*. The key heuristic notion there is that of a *function* of *anticommuting variables*, or more generally a function of m commuting and n and anticommuting variables. The way this is given a rigorous meaning depends on whether or not anticommuting numbers are used explicitly as well-defined objects and is therefore characteristic for two approaches. To understand this one notes first that an ordinary polynomial function of real or complex variables $x_1,...,x_m$ is given by an expression

$$f(x_1,...,x_m) = \sum_k \frac{1}{k!} \sum_{i_1,...,i_k=1,...,m} \alpha^{(k)}_{i_1...i_k} x_{i_1}...x_{i_k} \tag{1}$$

with the $\alpha^{(k)}_{i_1...i_k}$ real or complex coefficients, symmetric in the indices $i_l...i_k$. Antisymmetric coefficients would obviously make the function f identically 0. In physics and in particular in field theory one has found it nevertheless convenient to use expressions like (1) with antisymmetric coefficients. One then pretends that one still has non-trivial functions, or at least objects that can be handled like functions by assuming in an ad-hoc manner that the x_j have to be thought of as anticommuting. In the algebraic approach one starts from the fact that in the symmetric case the polynomial functions (1) form a commutive algebra, the *symmetric algebra* $\mathbb{S}_m$ over m generators. This means that f in (1) can be seen either as a function from $\mathbb{R}^n$ to $\mathbb{R}$ or $\mathbb{C}^n$ to $\mathbb{C}$ or alternatively as an element of the algebra $\mathbb{S}_m$. Operations on functions such as multiplication with a variable x_j and differentiation can be interpreted as algebraic operations on $\mathbb{S}_m$. In the antisymmetric case there is an analogue of the algebra $\mathbb{S}_m$, which is not commutive but graded-commutative, i.e. commutative up to minus signs between odd elements. This is Λ_n, the *antisymmetric algebra* (or Grassmann algebra) over n generators. Multiplication and differentiation as algebraic operations on $\mathbb{S}_m$ have direct analogues as operations on Λ_n. In the algebraic point of view a function of n anticommuting variables is not a function at all, i.e. not a map, but just an element of the algebra Λ_n. As such it can be written as a linear combination of basis elements, with real or antisymmetric coefficients $\alpha^{(k)}_{i_1...i_k}$. This expression (1) with fixed basis elements $f_{i_1} \wedge ... \wedge f_{i_k}$, f_j the generators of Λ_n, instead of products of variables $x_{i_1}...x_{i_k}$. Multiplication and differentiation are operations on the algebra Λ_n. In the geometric approach one retains for the antisymmetric case a true function picture through the device of introducing an unspecified but fixed Grassmann algebra $\mathfrak{B}$. The x_j are taken as variable elements of the odd part $\mathfrak{B}^{(1)}$ of $\mathfrak{B}$. Expression (1) with antisymmetric coefficients $\alpha_{i_1}...i^{(k)}_k$ then defines a polynomial function from $\mathfrak{B}^{(1)} \times ... \times \mathfrak{B}^{(1)}$ into $\mathfrak{B}$. This remains the case if one allows as more natural that the coefficients $\alpha^{(k)}_{i_1...i_k}$ also have values in $\mathfrak{B}$. In both pictures a function can be given by a set of antisymmetric coefficients $\alpha^{(k)}_{i_1...i_k}$. Because $\mathbb{R}$ or $\mathbb{C}$ is a subalgebra of $\mathfrak{B}$ there is an obvious identification of the collection of "functions" in the algebraic approach with a subset of functions in the geometric one. Our choice of the terms minimal and extended for algebraic and geometric on this level has to do with this fact. This will be explained in a more systematic manner in the second paper. All the foregoing can be generalized in an obvious way to the case of functions of m commuting and n anticommuting variables. One can also use for the dependence on the commuting variables C^∞ functions instead of polynomials. Again this will be discussed systematically and in a base independent manner in the next paper. The explicit formulas of the extended picture that we need in this paper can be found in the appendix.

III. The one-dimensional Wess-Zumino model

A. Basic formulas

The Wess-Zumino model as it was introduced in 1974 [1] is a supersymmetric field theory in four-dimensional space-time. It contains as true dynamical variables a scalar field $A(x)$, a pseudo-scalar field $B(x)$ and a Majorana spinor field $\psi(x)$. In addition to this there are two auxiliary fields, a scalar field $F(x)$ and a pseudo-scalar field $G(x)$. See also Refs. [3,4] and [5]. By suppressing the space variables x^j one is led to a much simpler model in which the fields depend only on a time variable $x^0=t$ and in which moreover the number of fields is reduced. Although we shall persist in speaking of fields and field theory, what we have in fact, in the classical version that we shall discuss first, is a simple but still non-trivial dynamical system, or rather because of the presence of anticommuting variables a superdynamical system according to Kupershmidt [26] or a pseudo-classical system in the language of Casalbuoni [27]. The quantum version belongs to quantum mechanics.

We give first the basic formulas of what we call the one-dimensional Wess-Zumino model and what also might be called the $N=2$ version of such a model. After this we shall discuss aspects of their precise mathematical meaning.

The model has dynamical fields $A(t), \psi_1(t)$ and $\psi_2(t)$. The three fields are real, $A(t)$ is commuting and the $\psi_j(t)$ are anticommuting, all this in a sense to be made precise. The $\psi_j(t)$ will be written together as $\psi(t)$ and in general a spinor notation will be used although at this one-dimensional level this has no group theoretical meaning. There are no auxiliary fields yet. The rôle of such fields will be made clear by first doing without them. The fields satisfy a system of non-linear *evolutions equations*

$$\left[\frac{d^2}{dt^2} + m^2\right]A = 3\lambda m A^2 - 2\lambda^2 A^3 - i\lambda\bar{\psi}\psi \tag{2}$$

$$\left[\gamma\frac{d}{dt} - m\right]\psi = -2\lambda A\psi \tag{3}$$

with γ the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; the adjoint $\bar{\psi}$ defined as $\bar{\psi}=\psi^T\gamma$, and consequently $\bar{\psi}\psi=\psi^T\gamma\psi=\Sigma_{j,k}\psi_j(\gamma)_{jk}\psi_k$; γ a real coupling constant; m a real parameter corresponding to a mass in higher dimension and to an oscillator frequency at this level.

These field equations are the Euler-Lagrange equations of a variational problem given by the Lagrangian

$$\mathfrak{L}\left[A, \frac{dA}{dt}, \psi, \frac{d\psi}{dt}\right] = \frac{1}{2}\left(\frac{dA}{dt}\right)^2 - \frac{1}{2}m^2A^2 -$$
$$-\frac{i}{2}\bar{\psi}\gamma\frac{d\psi}{dt} + \frac{i}{2}m\bar{\psi}\psi + i\gamma A\bar{\psi}\psi + \lambda m A^3 - \frac{1}{2}\lambda^2 A^4. \tag{4}$$

The model has symmetry under time translations and this is all that remains of Poincaré symmetry in four dimensions. This symmetry may be expressed by the statement that under variations $\delta A = -a\frac{dA}{dt}$, $\delta\psi = -a\frac{d\psi}{dt}$, with a a real parameter, the variation of $\mathfrak{L}$ is a total derivative

$$\delta\mathfrak{L} = -a\frac{d}{dt}\mathfrak{L} \tag{5}$$

Associated with this in the usual way is a conserved quantity, the energy

$$\frac{1}{2}(\frac{dA}{dt})^2 + \frac{1}{2}m^2A^2 - \frac{i}{2}m\bar{\psi}\psi + i\lambda A\bar{\psi}\psi - \lambda mA^3 + \frac{1}{2}\lambda^2A^4. \tag{6}$$

What makes the model interesting is the existence of an extra symmetry. For variations

$$\begin{aligned} \delta A &= i\bar{\psi}\epsilon \\ \delta\psi &= \frac{dA}{dt}\gamma\epsilon + (mA - \lambda A^2)\epsilon \end{aligned} \tag{7}$$

with $\epsilon = (\epsilon_1, \epsilon_2)$ real anticommuting parameters, the variation of the Lagrangian is again a total derivative

$$\delta\mathfrak{L} = \frac{i}{2}\frac{d}{dt}\left\{\frac{dA}{dt}\bar{\psi}\epsilon + (mA - \lambda A^2)\bar{\psi}\gamma\epsilon\right\}. \tag{8}$$

This expresses the *supersymmetry* of the model. There is a corresponding two-component conserved quantity, derived in the usual manner

$$\frac{dA}{dt}\psi + (mA - \lambda A^2)\gamma\psi. \tag{9}$$

An interesting question which we shall discuss in detail further on is whether this symmetry admits a non-infinitesimal formulation in terms of a proper symmetry group.

Note that it is essential for the model that the variable ψ is anticommuting. For commuting ψ the term in the Lagrangian (4) which couples A and ψ would drop out. Equations (2) and (3) become mutilated and (3) is no longer the variational equation connected with (6), and so on. The coherence between the formulas that are characteristic for the model is completely lost.

B. Mathematical aspects

Consider a complex Grassmann algebra $\mathfrak{B}$. It can be written as a direct sum $\mathfrak{B} = \mathfrak{B}^{(0)} \oplus \mathfrak{B}^{(1)}$, with $\mathfrak{B}^{(0)}$ and $\mathfrak{B}^{(1)}$ its even and odd parts. $\mathfrak{B}$ should have a conjugation $g \to g^*$, determined by a conjugation in the vectorspace by which $\mathfrak{B}$ is generated as exterior algebra, and with the usual properties $g^{**} = g$, $(\lambda g)^* = \lambda^* g^*$ for $\lambda \in \mathbb{C}$, $(g_1g_2)^* = g_2^*g_1^*$, etc. The conjugation leaves $\mathfrak{B}^{(0)}$ and $\mathfrak{B}^{(1)}$ separately invariant. $\mathfrak{B}_{sc}$, $\mathfrak{B}^{(0)}_{sc}$ and $\mathfrak{B}^{(1)}_{sc}$ are the "real", i.e. self-conjugate parts of $\mathfrak{B}$, $\mathfrak{B}^{(0)}$ and $\mathfrak{B}^{(1)}$. Eventually $\mathfrak{B}$ will turn out to be the unspecified but fixed Grassmann algebra of commuting and anticommuting variables that is characteristic for the extended picture as we sketched it in section II. The matter of its dimension, finite or infinite, has been discussed from various points of view, see e.g. Ref. [13]. Given the further developments that we have in mind the question is for us not very important. In this elementary part of our presentation we wish to avoid the technicalities of infinite dimensional Grassmann algebras, so we assume for the time being $\mathfrak{B}$ to be of finite dimension 2^N, with $N \geqslant 2$ in order to avoid a trivial situation.

Using $\mathfrak{B}$ we define the *fields* of the model as C^∞ functions

$$A: \ \mathbb{R} \rightarrow \mathfrak{B}_{sc}^{(0)}$$
$$\psi: \ \mathbb{R} \rightarrow \mathfrak{B}_{sc}^{(1)} \times \mathfrak{B}_{sc}^{(1)} \tag{10}$$

This takes care of the idea of the fields as "real" commuting and anticommuting objects and of (2) and (3) as well-defined differential equations. By choosing an appropriate basis in $\mathfrak{B}$ as a vectorspace equations (2) and (3) can of course be written as a set of $3 \cdot 2^{N-1}$ equations for a system of real-valued functions. This may in some respects be a useful mathematical picture to fall back on, although it is very much against the intuitive spirit of the subject.

The fields A and ψ, and their derivatives are functions with values in the Grassmann algebra $\mathfrak{B}$, but still dependent on the real variable t (or x^μ in higher dimension). The Lagrangian (4) and the way it is used to generate evolution equations and conserved quantities brings us to the Berezin calculus proper, the in first instance symbolic calculus for functions of anticommuting variables that is such a characteristic feature of supersymmetry. We have collected in an appendix the main formulas of the rigorous version of the Berezin calculus in what we call the extended interpretation, or at least the elementary part that is sufficient here. Using the concepts of this appendix it is not hard to give a precise meaning to most of the material sketched in the preceding part of this section. The Lagrangian (4) gives an action $\int_{t_a}^{t_b} \mathfrak{L} dt$, which is for every interval $[t_a, t_b]$ a functional of the fields A and ψ, with values in $\mathfrak{B}_{sc}^{(0)}$. If we mean by change under variations $\delta A, \delta\psi$ the first order effect in the real parameter α of the substitution $A(t) \rightarrow A(t) + \alpha A'(t)$, $\psi(t) \rightarrow \psi(t) + \alpha\psi'(t)$, with A' and ψ' fixed functions of t with values in $\mathfrak{B}_{sc}^{(0)}$, respectively $\mathfrak{B}_{sc}^{(1)} \times \mathfrak{B}_{sc}^{(1)}$, then the action gives in the usual way the evolution equations (2) and (3) as Euler-Lagrange equations of the form

$$\frac{\partial \mathfrak{L}}{\partial A} - \frac{d}{dt} \frac{\partial \mathfrak{L}}{\partial (\frac{dA}{dt})} = 0 \tag{11}$$

$$\frac{\partial \mathfrak{L}}{\partial \psi_j} - \frac{d}{dt} \frac{\partial \mathfrak{L}}{\partial (\frac{\partial \psi_j}{dt})} = 0, \quad j = 1,2. \tag{12}$$

The derivation of the conserved quantities (6) and (9) can be understood in a similar way. The finite dimensionality of $\mathfrak{B}$ leaves us with a few minor but characteristic gaps. One derives for instance

$$\int_{t_a}^{t_b} \sum_{j=1,2} \psi'_\gamma \left[\frac{\partial \mathcal{L}}{\partial \psi_\gamma} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \psi_\gamma} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial (\frac{d\psi'_j}{dt})} \right] dt = 0, \ \forall \psi'. \tag{13}$$

This condition is satisfied by equation (12), however strictly speaking (13) does not imply (12), due to the presence of nilpotent elements.

One should finally observe that formulas like (11) and (12) are still rather hybrid because ordinary derivatives with respect to t appear side by side with Berezin derivatives. This is connected with the fact that the formulation of supersymmetric theory in this section is still an intermediate one. The full potentialities of the Berezin calculus in supersymmetry will be realized in the superspace-superfield formulation which is more concise and elegant but also harder to make rigorous. We shall discuss this in section V.

IV. Supersymmetry as a group symmetry. The role of Auxiliary fields

A. Variations and 1-parameter groups

In field theory symmetry groups such as the Poincaré group act in the first place at the classical level by transformations of the fields as functions on space-time and secondly at the quantum level by unitary operators in the Hilbert space of quantum states. It is standard practice in field theory to discuss symmetry in infinitesimal form, which means in first instance the language of variations. Supersymmetry is no exception as it was originally introduced in this manner, without any reference to a symmetry group as such.

We take therefore as starting point for the the discussion of supersymmetry in our model the variations of the fields A and ψ given by the formulas (7). We shall first exhibit the full group action generated by the variations as infinitesimal transformations. From this the "abstract" group is found. It is the supersymmetry group of the model and contains as a subgroup the one-dimensional time-translation group, the remnant of the Poincaré group in this situation.

In a precise formulation variations δA and $\delta\psi$ stand for the first order part of a one-parameter set of transformations of the functions A and ψ

$$\begin{aligned} A_\alpha &= A + \alpha(i\bar{\psi}\epsilon) + \ldots \\ \psi_\alpha &= \psi + \alpha(\frac{dA}{dt}\gamma + mA - \gamma A^2)\epsilon + \ldots \end{aligned} \tag{14}$$

in which α is a real parameter and ϵ is for the time being kept fixed as an element of $\mathfrak{B}_{sc}^{(1)} \times \mathfrak{B}_{sc}^{(1)}$. Denoting for the moment the pair (A,ψ) as ϕ, this can be written as

$$\phi_\alpha = \phi + \alpha L(\phi) + \ldots \tag{15}$$

with $L(\phi)$ a *non-linear* operator in the space of functions $\phi(t)=(A(t),\psi(t))$. The one-parameter set of transformations $\phi \to \phi_\alpha$ should moreover be a one-parameter *group*. This means that ϕ_α must be solution of the differential equation

$$\frac{d}{d\alpha}\phi_\alpha = L(\phi_\alpha). \tag{16}$$

In principle such an ordinary differential equation can be expected to have a solution of some sort although at this formal infinite dimensional stage no precise statement can be made. Due to the non-linearity of L there is certainly no explicit solution of (16) at hand.

It is possible to improve matters considerably by the introduction of an additional variable. So-called *auxiliary fields* are a general feature of supersymmetric field theory and several reasons for their presence can probably be given. In any case the rôle of such fields in making the group aspects of supersymmetry more transparant will be clear from what follows.

B. The one-dimensional model with an auxiliary field

We extend the model as given in section III part A by an additional commuting field F, i.e. a C^∞ function

$$F:\mathbb{R}\rightarrow\mathfrak{B}_{sc}^{(0)}. \tag{17}$$

Instead of (2) and (3) we have the equations

$$\frac{d^2A}{dt^2} + (m-2\lambda A)F + i\lambda\bar{\psi}\psi = 0. \tag{18}$$

$$\frac{d\psi}{dt} + (m-2\lambda A)\gamma\psi = 0. \tag{19}$$

$$F - mA + \lambda A^2 = 0. \tag{20}$$

These come from a Lagrangian

$$\mathfrak{L}(A, \frac{dA}{dt}, \psi, \frac{d\psi}{dt}, F) = (\frac{dA}{dt})^2 + \frac{1}{2}F^2 -$$

$$-(mA-\lambda A^2)F - \frac{i}{2}\bar{\psi}\gamma\frac{d\psi}{dt} + \frac{i}{2}(m-2\lambda A)\bar{\psi}\psi. \tag{21}$$

The energy as constant of the motion becomes

$$\frac{1}{2}(\frac{dA}{dt})^2 - \frac{1}{2}F^2 + (mA-\lambda A^2)F - \frac{i}{2}(m-2\lambda A)\bar{\psi}\psi. \tag{22}$$

Supersymmetry now means that for variations

$$\begin{aligned} \delta A &= i\bar{\psi}\epsilon \\ \delta\psi &= \frac{dA}{dt}\gamma\epsilon + F\epsilon \\ \delta F &= -i\overline{\frac{d\psi}{dt}}\gamma\epsilon \end{aligned} \tag{23}$$

the Lagrangian (21) changes in first order as

$$\delta\mathfrak{L} = \frac{i}{2}\frac{d}{dt}\left\{\frac{dA}{dt}(\bar{\psi}\epsilon) - (F-2mA+2\lambda A^2)\bar{\psi}\gamma\epsilon\right\} \tag{24}$$

and that there is a corresponding conserved quantity

$$\frac{dA}{dt}\psi + (mA - \lambda A^2)\gamma\psi. \tag{25}$$

Note that (20) is an algebraic equation. By using it the field F can be eliminated and one recovers the earlier formulas. The model in this formulation has the same dynamical content, but there is an improvement in the formulation of the supersymmetry aspects: The supersymmetry variations (23) are *linear* in the fields.

C. The action of the supersymmetry group on the fields

If ϕ now stands for the triple (A,ψ,F) the action of the one-parameter supersymmetry groups given by the variations (23) is again described by the differential equation (16) in which, however, L has become a *linear* operator. Consequently there is a formal solution

$$\phi_\alpha = e^{\alpha L}\phi_0 \tag{26}$$

which has in fact a rigorous meaning and can be written in explicit form because there are only a finite number of terms in the exponential series, due to the presence of elements ϵ_j from $\mathfrak{B}^{(1)}$. We write $L(\epsilon)$ to indicate the dependence on ϵ, and absorb in this ϵ the real parameter α. The variations (23) can be written as

$$L(\epsilon)(A,\psi,F) = (i\bar{\psi}\epsilon,\ \frac{dA}{dt}\gamma\epsilon + F\epsilon,\ -i\frac{d\bar{\psi}}{dt}\gamma\epsilon). \tag{27}$$

A simple calculation then gives the action of $e^{L(\epsilon)} = 1 + L(\epsilon) + 1/2L(\epsilon)^2$ on the fields as

$$e^{L(\epsilon)}(A,\psi,F) = (A',\psi',F') \tag{28}$$

with

$$A' = A + i\bar{\psi}\epsilon + \frac{i}{2}F\bar{\epsilon}\epsilon \tag{29}$$

$$\psi' = \psi + \left[\frac{dA}{dt}\gamma\epsilon + F\right]\epsilon - \frac{i}{2}\gamma\frac{d\psi}{dt}\bar{\epsilon}\epsilon$$

$$F' = F - i\frac{d\bar{\psi}}{dt}\gamma\epsilon - \frac{i}{2}\frac{d^2A}{dt^2}\bar{\epsilon}\epsilon.$$

The set of "pure" supersymmetry transformations $e^{L(\epsilon)}$ is in itself not a group. This is of course well-known and is clear from the fundamental commutation relation which is an easy consequence of (27)

$$[L(\epsilon),L(\epsilon')] = 2i(\bar{\epsilon}\gamma\epsilon')\frac{d}{dt} \tag{30}$$

$$\forall \epsilon,\epsilon' \mathfrak{B}^{(1)}_{sc} \times \mathfrak{B}^{(1)}_{sc}.$$

The presence of $\frac{d}{dt}$ on the right-hand side shows the link with time translations, and gives a precise meaning to the standard heuristic statement that two subsequent supersymmetry transformations add up to a translation in space-time. If we denote $-\frac{d}{dt}$ as P we can extend (30) in a trivial way to

$$\begin{aligned} [L(\epsilon),L(\epsilon')] &= -2i(\bar{\epsilon}\gamma\epsilon')P \\ [L(\epsilon),P] &= P,P] = 0 \end{aligned} \tag{31}$$

which is the set of commutation relations for Lie algebra over $\mathbb{R}$, of dimension $3 \cdot 2^{N-1}$, and also over the commutative algebra $\mathfrak{B}_{sc}(0)$. The group associated with this Lie algebra is the full supersymmetry group generated by the $e^{L(\epsilon)}$ and containing the time translations as a one-parameter subgroup. We write the time translations as $e^{\alpha P}$, for $\alpha \in \mathbb{R}$ with the obvious meaning $(e^{\alpha P}\phi)(t) = \phi(t-\alpha)$, for every C^∞ function $\phi:\mathbb{R}\rightarrow\mathfrak{B}$. Because of the occurrence of even elements of $\mathfrak{B}$ in the right-hand side of (30), we need to give meaning to the more general exponential e^{aP}, for $a \in \mathfrak{B}_{sc}^{(0)}$. For this we note that every such a can be uniquely decomposed as

$$a = a_b + a_s \tag{32}$$

with a_b a real number, i.e. a real multiple of the identity element of $\mathfrak{B}$, and A_s an element of $\mathfrak{B}_{sc}^{(0)}$ which is nilpotent due to the finite dimension of $\mathfrak{B}$. (The subscripts b and s stand for *body* and *soul*, a terminology introduced by DEWITT [20]). The decomposition (32) enables us to define a^{aP} as $e^{aP} = e^{a_b P} e^{a_s P} = e^{a_s P} e^{A_b P}$ with

$$(e^{a_s P}\phi)(t) = \sum_k \frac{(-1)^k}{k!} a_s^k \frac{d^k}{dt^k}\phi(t). \tag{33}$$

This is a satisfactory definition rigorously meaningful for every C^∞ function ϕ, because due to the nilpotency of a_s the series breaks off after a finite number $(k > \frac{N}{2})$ of terms. All this together leads us to definition of the general supersymmetry transformation as

$$W(a,\epsilon) = e^{aP+L(\epsilon)} = e^{aP} e^{L(\epsilon)} e^{aP} \tag{34}$$

$$\forall a \in \mathfrak{B}_{sc}^{(0)}, \quad \epsilon \in \mathfrak{B}_{sc}^{(1)} \times \mathfrak{B}_{sc}^{(1)}$$

Using (34) and either the explicit expressions (28), (29) and (33) or the commutation relations (31) combined with the Baker-Cambell-Hausdorff formula one derives easily a multiplication formula

$$W(a,\epsilon)W(a',\epsilon') = W(a+a'-i\bar{\epsilon}\gamma\epsilon', \ \epsilon+\epsilon') \tag{35}$$

$$\forall a,a' \in \mathfrak{B}_{sc}^{(0)}, \quad \epsilon,\epsilon' \in \mathfrak{B}_{sc}^{(1)} \times \mathfrak{B}_{sc}^{(1)}$$

which means that the $W(a,\epsilon)$ form indeed a group. This is the *full supersymmetry group* generated by the Lie algebra (31), containing on one hand as a subgroup the usual time translations acting separately on the fields and on the other hand the "pure" supersymmetry transformations mixing even and odd fields.

D. The supersymmetry group as an "abstract" group

The action of the supersymmetry operators $W(a,\epsilon)$ on the fields, as given by formulas (28), (20), (33) and (34), is fairly complicated and not very transparant. It represents, however, a group which can be described explicitly in a very simple manner. We denote this group, the "abstract" supersymmetry group of the problem, as $\mathcal{G}$. It consists obviously of all pairs (a,ϵ), with $a \in \mathcal{B}_{sc}^{(0)}$, $\epsilon \in \mathcal{B}_{sc}^{(1)} \times \mathcal{B}_{sc}^{(1)}$ with group multiplication defined as

$$(a,\epsilon)(a',\epsilon') = (a + a' - i\bar{\epsilon}\gamma\epsilon',\ \epsilon + \epsilon'). \tag{36}$$

The unit element is of course (0,0) and the inverse $(a,\epsilon)^{-1}$ is $(-a,-\epsilon)$. $\mathcal{G}$ is a $3{\cdot}2^{N-1}$-dimensional real Lie group. In the true spirit of the subject one prefers to think of $\mathcal{G}$ as a group described by one commuting and two anticommuting and two anticommuting parameters, in fact a (1,2)-dimensional *supergroup.* A rigorous definition of this concept has first been given by ROGERS [28], following earlier somewhat different and more formal ideas of BEREZIN and KAC [29]. There is however no need to invoke all this for this elementary and explicit example. (An even simpler example has been briefly discussed by LANGOUCHE and SCHÜCKER [39]).

V. SUPERSPACE AND SUPERFIELDS

A. Introductory remarks

The superspace-superfield formalism may be regarded as a convenient way of writing the separate fields of a supersymmetric theory, in our case A, and F, as a single field-like object. This superfield then depends not only on the space-time variables x^{μ} - in our case only t - but also on a set of additional anticommuting variables θ_j. The original fields appear as coefficients in the expansion of the superfields with respect to the θ_j. The result of this rewriting is not just a reduction in the number of fields. The theory as a whole becomes in almost every respect simpler and above all formally more elegant and natural, to such an extent that one feels that the superfield formulation is more than a book-keeping device but represents in some way the mathematical essence of supersymmetry. This becomes even more apparent if one follows the group-theoretical motivation for the definition of superfields that the inventors of the formalism, SALAM and STRATHDEE, gave in their first short paper on the subject [31], and which they curiously enough omitted in subsequent more detailed publications [32, 33]. Their argument runs as follows: The relativistic (classical) fields of various types can be found in a systematic way by constructing the representations of the Poincaré $\mathcal{P}$, induced by the finite dimensional representations of the homogeneous Lorentz group $\mathcal{L}$, or its covering group $SL(2,\mathbb{C})$, as a subgroup of $\mathcal{P}$. The fields emerge as functions, or more general as sections of vector bundles, defined on the coset space $\mathcal{P}/\mathcal{L}$. This coset space can be identified with Minkowski space-time. Supersymmetry assumes the existence of a larger symmetry group $\mathcal{G}$, the super Poincaré group. It is then natural to consider representations induced from $\mathcal{L}$ to $\mathcal{G}$. As a result one gets superfields defined on the coset space $\mathcal{G}/\mathcal{L}$. This coset space can be regarded as an extension of Minkowski space and is called superspace.

B. Superspace and superfields in one dimension

In our one-dimensional version of the Wess-Zumino model the situation is much simpler, but the ideas of superspace and superfields retain their essential features. The homogeneous Lorentz group is reduced to the identity, so $\mathcal{G}/\mathcal{L}$ is $\mathcal{G}$ itself, and the induced representation is what usually is called the left regular representation of $\mathcal{G}$ defined in terms of functions on $\mathcal{G}$.

Let the supersymmetry group $\mathcal{G}$ of the model be given as in chapter IV, i.e. as the set of pairs (a,ϵ), $a\in\mathfrak{B}_{sc}^{(1)}\times\mathfrak{B}_{sc}^{(1)}$, with multiplication rule (36). Let $\mathfrak{C}_{sc}^{(0)}$ be the space of functions $\phi:\mathcal{G}\rightarrow\mathfrak{B}_{sc}^{(0)}, C^{\infty}$ as a function of one commuting and two anticommuting variables in the sense discussed in the appendix. The left action of $\mathcal{G}$ on itself induces a representation of $\mathcal{G}$ by linear transformations $\hat{W}(a,\epsilon)$ in $\mathfrak{C}_{sc}^{(0)}$ according to

$$(\hat{W}(a,\epsilon)\phi)(\tau,\theta) := \phi(a,\epsilon)^{-1}(\tau,\theta) = \phi(\tau-a+i\bar{\epsilon}\gamma\theta,\ \theta-\epsilon). \tag{37}$$

One writes $\hat{W}(a,\epsilon)=\hat{W}(a,0)\hat{W}(0,\epsilon)$ and obtains by expanding (37) separately for $\hat{W}(a,0)=e^{a\hat{P}}$ and $\hat{W}(0,\epsilon)=e^{L(\epsilon)}$ the action of $\hat{P}$ and $\hat{L}(\epsilon)$ as Berezin differential operators in $\mathfrak{C}_{sc}^{(0)}$

$$\begin{aligned}\hat{P} &= -\frac{\partial}{\partial\tau}\\ L(\epsilon) &= i\bar{\epsilon}\gamma\theta\frac{\partial}{\partial\tau}-\sum_{j=1,2}\epsilon_j\frac{\partial}{\partial\theta_j}.\end{aligned} \tag{38}$$

These operators satisfy of course the Lie algebra commutation relations (31). We expand an arbitrary superfield $\phi(\tau,\theta)$ in powers of the variables θ_j in the following manner

$$\phi(\tau,\theta) = A(\tau)-i\overline{\psi(\tau)}\theta + \frac{i}{2}F(\tau)\bar{\theta}\theta. \tag{39}$$

Note that the A,ψ and F in this formula are not yet the fields A,ψ and F of the previous chapters because they are C^{∞} functions of $\tau\in\mathfrak{B}_{sc}^{(0)}$. (Due to a proper choice of factors i they are "real" i.e. have self conjugate values. They have also the right commutation or anticommutation properties). The operator $\hat{P}$ acts on the coefficients (or components) as $-\frac{d}{dt}$. For $\hat{L}(\epsilon)$ one finds easily

$$L(\epsilon)\phi = A'-i\bar{\psi}'\theta + \frac{i}{2}F'\bar{\theta}\theta \tag{40}$$

with

$$\begin{aligned}A' &= i\bar{\psi}\epsilon\\ \psi' &= \frac{dA}{d\tau}\gamma\epsilon+F\epsilon\\ F' &= -i\frac{d\bar{\psi}}{d\tau}\gamma\epsilon\end{aligned} \tag{41}$$

There is a $1-1$ correspondence between C^∞ functions $f:\mathbb{R}\to\mathfrak{B}$ and C^∞ functions $\hat{f}\mathfrak{B}_{sc}^{(0)}\to\mathfrak{B}$, which we describe int he appendix where it is in fact used to give a proper definition of C^∞ functions of variables in $\mathfrak{B}_{sc}^{(0)}$. The correspondence is linear and carries $\frac{d}{dt}$ over in $\frac{d}{d\tau}$. If we regard A,ψ and F in (39) as the $\hat{A},\hat{\psi}$ and $\hat{F}$ connected with the A,ψ and F from the previous chapters it is then clear that (41) enables us to identify the representation of the supersymmetry group and its Lie algebra in terms of superfields and given by (37) and (38) with the original representation in terms of separate ordinary fields as discussed in chapter IV. We shall henceforth write $W(a,\epsilon)$ for $W(a,\epsilon)$, etc.

C. Dynamical aspects

The kinematics of the model, i.e. the description in terms of fields and their transformation rules is greatly simplified by the introduction of $\phi(\tau,\epsilon)$. This is even more so for the dynamical aspects. The system of evolution equations (18), (19) and (20) for the components A,ψ and F can be collected into a single equation for ϕ

$$(T-m)\phi + \lambda\phi^2 = 0 \tag{42}$$

with the second order Berezin partial differential operator

$$T = -i\theta_1\theta_2\frac{\partial^2}{\partial\tau^2} + \theta_1\frac{\partial^2}{\partial\tau\partial\theta_2} - \theta_2\frac{\partial^2}{\partial\tau\partial\theta_1} + i\frac{\partial^2}{\partial\theta_1\partial\theta_2}. \tag{43}$$

The equivalence of (42) with (18), (19) and (20) is easily verified by writing (42) in components, and using again the $1-1$ correspondence $\hat{A}(\tau)\leftrightarrow A(t)$, $\frac{d}{d\tau}\leftrightarrow\frac{d}{dt}$, etc. It is also not hard to see that T and therefore equation (42) is invariant under supersymmetry transformations. In this model T seems to be an effective but not very enlightening expression. In the higher dimensional situation T will emerge however as a unique invariant operator within a natural differential geometric setting. The field equation (42) can be associated with a Lagrangian density in superspace. Denote the partial derivatives $\frac{\partial}{\partial\tau}, \frac{\partial}{\partial\theta_1}$, and $\frac{\partial}{\partial\theta_2}$ as ∂_0, ∂_1 and ∂_2 and write

$$\mathfrak{L}^s(\phi,\partial_0\phi,\partial_1\phi,\partial_2\phi,\theta_1,\theta_2) =$$
$$\frac{i}{2}(i\theta_1\partial_0\phi-\partial_1\phi)(i\theta_2\partial_0\phi-\partial_2\phi) + \frac{1}{2}m\phi^2 - \frac{1}{3}\lambda\phi^3. \tag{44}$$

Equation (42) is then indeed the Euler-Lagrange equation

$$\frac{\partial\mathfrak{L}^s}{\partial\phi} - \sum_{\alpha=0,1,2}\partial_\alpha\frac{\partial\mathfrak{L}}{\partial(\partial_\alpha\phi)} = 0. \tag{45}$$

One checks moreover that the original Lagrangian (21) in the component field picture can be recovered from $\mathfrak{L}^s$ as $i\int d\theta_2 d\theta_1\mathfrak{L}^s$.

Supersymmetry of the theory can be expressed in variational language. The infinitesimal action of the supersymmetry group as given by formula (38) is written as a variation

$$\delta\phi = (-a+i\bar{\epsilon}\gamma\theta)\partial_0\phi - \sum_{j=1,2} \epsilon_j\partial_j\phi. \tag{46}$$

Insertion of this $\delta\phi$ in $\mathcal{L}^s$ leads to a variation $\delta\mathcal{L}^s$ which after a fairly extensive but elementary calculation is found to have the following simple form as a divergence in superspace

$$\delta\mathcal{L}^s = \sum_{\alpha=0,1,2} \partial_\alpha B_\alpha \tag{47}$$

with

$$\begin{aligned} B_0 &= (-a+i\bar{\epsilon}\gamma\theta)\mathcal{L}^s \\ B_j &= \epsilon_j\mathcal{L}^s,\ \ j=1,2. \end{aligned} \tag{48}$$

This implies the existence of a conserved superspace current. Combining (47) and (48) with the general variation formula

$$\delta\mathcal{L}^s = \delta\phi\left[\frac{\partial\mathcal{L}^s}{\partial\phi} - \sum_{\alpha=0,1,2}\partial_\alpha\frac{\partial\mathcal{L}^s}{\partial(\partial_\alpha\phi)}\right] + \sum_{\alpha=0,1,2,}\partial_\alpha\left[\delta\phi\frac{\partial\mathcal{L}^s}{\partial(\partial_\alpha\phi)}\right] \tag{49}$$

one finds that for solutions ϕ of (45) or equivalently (42)

$$\sum_{\alpha=0,1,2}\partial_\alpha J_\alpha = \sum_{\alpha=0,1,2}\partial_\alpha\left[\delta\phi\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} - B_\alpha\right] = 0. \tag{50}$$

More explicitly one obtains for J_0 the expression

$$\begin{aligned} J_0 = {} & \frac{ia}{2}\{(\partial_0\phi)^2\theta_1\theta_2 + \partial_1\phi\partial_2\phi\} + \sum_{j=1,2} i\epsilon_j(\partial_0\phi\partial_j\phi)\theta_1\theta_2 + \\ & + (a-i\bar{\epsilon}\gamma\theta)(\frac{1}{2}m\phi^2 - \frac{1}{3}\lambda\phi^3). \end{aligned} \tag{51}$$

There is again a connection with the conserved quantities (22) and (25) of the earlier component field picture. Calculation of the Berezin integral $i\int d\theta_2 d\theta_1 J^0$ gives, up to an irrelevant overall minus sign, precisely the sum of the expressions (22) and (25), the energy and the conserved quantity connected with supersymmetry proper, in terms of the fields A, ψ and F.

For the sake of completeness one may also calculate J_1 and J_2 as

$$\begin{aligned} J_1 = {} & -\frac{a}{2}\partial_0\phi(\partial_0\phi\theta_2 + i\partial_2\phi) + \epsilon_1\partial_1\phi(\partial_0\phi\theta_2 - i\partial_2\phi) \\ & - \epsilon_2\partial_0\phi\partial_2\phi\theta_2 - \epsilon_1(\frac{1}{2}m\phi^2 - \frac{1}{3}\lambda\phi^3). \end{aligned} \tag{52}$$

$$\begin{aligned} J_2 = {} & \frac{a}{2}\partial_0\phi(\partial_0\phi\phi\theta_1 + i\partial_1\phi) + \epsilon_1\partial_0\phi\partial_1\phi\theta_1 + \epsilon_2\partial_2\phi(\partial_0\phi\theta_1 + \partial_1\phi) \\ & - \epsilon_2(\frac{1}{2}m\phi^2 - \frac{1}{3}\gamma\phi^3) \end{aligned} \tag{53}$$

and verify explicitly that indeed (50) is satisfied.

Part of the foregoing discussion cannot of course be considered as mathematically rigorous. In particular there has not yet been given a proper mathematical formulation of the underlying variational calculus in superspace based on a superspace action $\int d\tau d\theta_1 d\theta_2 \mathcal{L}^s$, well defined with due attention to behaviour at the boundary of the region of integration. Rigorous work has been done on Berezin integration, see e.g. Refs. [34] and [35] but most of this is rather general and it is not clear whether it can be used to set up the sort of variational formalism that would be needed here. Nevertheless all this is very suggestive and strongly supports the point of view that the language of superfields and superspace is essential for supersymmetry. A final observation which stresses this even further is the following: So far we have followed the rather loose standard practice of speaking of a symmetry when a variation of the Lagrangian has the form of a total derivative or in higher dimension a divergence. This implies the existence of a conserved quantity or current. For the purpose of quantization one needs however a stronger definition of symmetry, namely invariance of the action. This is particularly obvious when thinking in terms of path integral quantization. For variations of the fields that are connected with variations in the independent variable or variables, invariance of the action means in normal variational theory not only that $\delta\mathcal{L}$ is a total derivative or divergence but also that these have the specific form

$$\delta\mathcal{L} = -\frac{d}{dt}\{\delta t \mathcal{L}\} \tag{54}$$

in the one-dimensional case and

$$\delta\mathcal{L} = -\frac{\partial}{\partial x^\mu}\{\delta x^\mu \mathcal{L}\} \tag{55}$$

in higher dimensions.

Inspection of our model in the component field form, with and without auxiliary field F, shows that for time translations $\delta t = a$ one has of course $\delta\mathcal{L} = -a\mathcal{L}$, which is indeed (54) but that for the supersymmetry variations (23) and (7) one has the expressions (24) and (8) for the variations $\delta\mathcal{L}$ which are total derivatives, but fail to be of the form (54). In the superspace formulation the variation (46) of the superfield ϕ is connected with variations $\delta_\tau = -a + i\bar{\epsilon}\gamma\theta$ and $\delta\theta\epsilon$ of the independent variables and resulting variation (47) of $\mathcal{L}^s$ does have a form which is a direct analogue of (55). This means that with respect to subsequent quantization the superspace-superfield formulation is the proper one for this classical supersymmetric model.

VI. Concluding Remarks

In this first paper we have discussed a very simple but nevertheless typical model of a supersymmetric field theory with the purpose of establishing basic concepts useful for our further study of the mathematics of supersymmetry and anticommuting variables. In presenting the model we have followed a line of argument which is a reduced version of a standard reasoning for the four-dimensional found in the physical literature. In this one introduces supersymmetry as a new symmetry in a rather ad-hoc manner: A certain set of boson and fermion fields can be transformed infinitesimally into each other by ingeniously chosen transformations and a Lagrangian can be found which remains invariant. None of this, the particular choice for the fields, the form of the transformations or the expression for the Lagrangian, is simple or obvious. Looking back one may find it quite astonishing that supersymmetry was ever discovered in this way. All this is of course particularly true for the four-dimensional theory, with more fields, the complications of the Poincaré group, etc. It is in the superspace formalism that the model and its properties become suddenly transparant and natural. The following may serve as a further illustration of this: The Wess-Zumino model is not an isolated case but it represents a whole class of supersymmetric classical field theories. The same is true for the one-dimensional version. This can be seen quite easily in the superspace formulation, where the term $\frac{1}{3}\gamma\phi^3$ in the superspace Lagrangian (44) can be immediately generalized to an arbitrary "superpotential" $V(\phi)$, which is a C^∞ function of ϕ in the sense given in the appendix. Instead of (42) one has now as Euler-Lagrangian equation the field equation

$$(T-m)\phi + V^{(1)}(\phi) = 0 \tag{56}$$

where we write generally $V(k)(\phi)$ for the k^{th} derivative of $V(\phi)$. This simple, situation which is obviously still supersymmetric, can be translated back into the component field picture in which we began the discussion of supersymmetry in section III of this paper. One writes ϕ again as $A-i\bar{\psi}\theta+\frac{i}{F}\bar{\theta}\theta$. One derives the following useful expansion formula for a C^∞ function of ϕ.

$$f(\phi) = f(A)-\mathrm{i}f^{(1)}(A)\bar{\psi}\theta+\frac{i}{2}(f^{(1)}(A)F-\frac{i}{2}f^{(2)}(A)\bar{\psi}\psi)\bar{\theta}\theta \tag{57}$$

and uses it together with the results from section V to rewrite (56) in terms of A,ψ and F. Finally one eliminates F and obtains as generalization of the evolution equations (2) and (3) from section III

$$\frac{d^2A}{dt} + m^2A = m(V^{(1)}(A)+AV^{(2)}(A))-V^{(1)}(A)V^{(2)}(A)-$$

$$-\frac{i}{2}V^{(3)}(A)\bar{\psi}\psi \tag{58}$$

$$(\gamma\frac{d}{dt}-m)\psi= -V^{(2)}(A)\psi. \tag{59}$$

The corresponding Lagrangian, the generalization of (4), is obtained by working out the Berezin integral $i\int d\theta_2 s\theta_1\mathfrak{L}$ and is then found to be

$$\mathfrak{L}(A,\frac{dA}{dt},\psi,\frac{d\psi}{dt}) = \frac{1}{2}(\frac{dA}{dt})^2-\frac{1}{2}m^2A^2-\frac{i}{2}\bar{\psi}\gamma\frac{d\psi}{dt}+\frac{i}{2}m\bar{\psi}\psi$$

$$-\frac{1}{2}V^{(1)}(A)^2 + mAV^{(1)}(A)-\frac{i}{2}V^{(2)}(A)\bar{\psi}\psi. \tag{60}$$

The supersymmetry variations (7) which are dependent on the potential V become

$$\begin{aligned} \delta A &= i\bar{\psi}\epsilon \\ \delta\psi &= \frac{dA}{dt}\gamma\epsilon + (mA - V^{(1)}(A))\epsilon. \end{aligned} \tag{61}$$

This completes the description of the more general class of supersymmetric models in the component field language. The loss of transparancy is evident. This again supports our general point of view that the idea of superspace is central in supersymmetry. The underlying reason for that is group theory or more precisely Lie algebra and theory. This will be developed more extensively in the next paper.

The mathematical framework for superdifferential geometry in the minimal/algebraic approach i.e. in terms of graded manifolds is well established and can be found in basic papers on the subject such as those by LEITES [22] and KOSTANT [23]. Seen from physics this approach is however less obvious than the extended/geometric picture. There is a paper by DELL and SMOLIN [36], but apart from this no further serious attempts seem to have been made to apply it explicitly to supersymmetric field theory. The mathematically oriented literature is underdeveloped in this respect. In our next paper we shall show that the extended version of the one-dimensional Wess-Zumino model, such as is given in this paper, contains as a skeleton an algebraic version of the same situation, i.e. a version from which the unspecified Grassmann algebra $\mathfrak{B}$ has been removed. This will be put in the context of a general algebraic scheme for supersymmetric field theory.

VII. APPENDIX

The heart of the heuristic method of anticommuting variables is differentiation and integration. This was largely invented by BEREZIN [37] and it is therefore proper to call it *Berezin calculus*. It can be understood rigorously in two distinct ways as we have discussed in section II. In this appendix we give the basic formulas for what we have called the *extended* approach. We restrict ourselves to the elementary part that we need in this paper.

Let $\mathfrak{B}$ be the fixed finite-dimensional complex Grassmann algebra that is typical for the extended formalism. We define a *polynomial* in m commuting and n anticommuting variables as an expression of the form

$$f(x_1,\ldots,x_m,\ \theta_1,\ldots,\theta_n = \\ = \sum_{p,q=0}^{\infty} \frac{1}{p!q!} \sum_{\substack{i_1,\ldots,i=1,2,\ldots,m \\ j_1,\ldots,j_q=1,2,\ldots,n}} x_{i_1}\ldots x_{i_m}\theta_{j_1}\ldots\theta_{j_n} a^{(p,q)}_{i_1\ldots i_p\, j_1\ldots j_q} \tag{62}$$

The coefficients $a^{(p,q)}_{i_1\ldots i_p\, j_1\ldots j_q}$ of which only a finite number are different from zero have values in $\mathfrak{B}$, are symmetric in the indices $i_1\ldots i_p$ and antisymmetric in $j_1\ldots j_q$. (Writing the variables x_j and in particular θ_j in front of the coefficients is a convention which keeps down the number of minus signs in what follows).

With the x_i and the θ_j variable elements of $\mathcal{B}^{(0)}$, respectively $\mathcal{B}^{(1)}$, expression (62) gives a *function,* i.e. a map

$$f: \underbrace{\mathcal{B}^{(0)}\times\ldots\times\mathcal{B}^{(0)}}_{m \text{ times}} \times \underbrace{\mathcal{B}^{(1)}\times\ldots\times\mathcal{B}^{(1)}}_{n \text{ times}} \to \mathcal{B} \tag{63}$$

Because of the finite dimension of $\mathcal{B}$ and the presence of nilpotent elements a single function in this sense can be represented by different polynomial expressions. To avoid inconsistencies in the further developments we think of functions $f(x_1,...,x_m, \theta_1,...,\theta_n)$ in the first place as expressions (62) given by a set of coefficients $a^{(p,q)}_{i_1...i_p\, j_1...j_q}$. This anticipates already the more explicitly algebraic point of view that we shall develop in the next paper.

A polynomial can be expanded in a finite Taylor series around a fixed set of values $(x_1,...,x_m, \theta_1,...,\theta_n)$

$$\begin{aligned} f(x_1+\nu_1,...,x_m+\nu_m, \theta_1+\epsilon_1,...,\theta_n+\epsilon_n) &= f(x_1,...,x_m, \theta_1,...,\theta_n) + \\ &+ \sum_{i=1}^{m} \nu_i f_i^{(1,0)}(x_1,...,x_m, \theta_1,...,\theta_n) + \\ &+ \sum_{j=1}^{m} \epsilon_j f_j^{(0,1)}(x_1,...,x_m, \theta_1,...,\theta_n) + \ldots . \end{aligned} \tag{64}$$

The polynomials $f_i^{(0,1)}$ and $f_j^{(1,0)}$ are by definition the *partial derivatives* of f

$$\frac{\partial f}{\partial x_i} := f_i^{(1,0)} \quad \frac{\partial f}{\partial \theta_j} := f_j^{(0,1)}. \tag{65}$$

Of course $\frac{\partial f}{\partial \theta_j}$ will in fact no longer depend on θ_j. Note that the usual definition of a partial derivative in standard calculus would be meaningless, e.g. for $\frac{\partial f}{\partial \theta_1}$ as

$$\lim_{\nu_1\to 0} \frac{f(x_1,...,x_m, \theta_1+\nu,...,\theta_n) - f(x_1,...,x_m, \theta_1,...,\theta_n)}{\nu_1} \tag{66}$$

Note also that our conventions are such that the coefficients in (62) are precisely the higher partial derivatives of f in 0

$$a^{(p,q)}_{i_1...i_p\, j_1...j_q} = \frac{\partial^{p+1}}{\partial x_{i_1}...\partial x_{i_p}\, \partial\theta_{j_1}...\partial\theta_{j_q}} f(0,...,0,...,0) \tag{67}$$

The partial differentiations commute or anticommute in a well-known manner

$$\begin{aligned} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} &= \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i} \\ \frac{\partial}{\partial x_i}\frac{\partial}{\partial \theta_k} &= \frac{\partial}{\partial \theta_k}\frac{\partial}{\partial x_i} \\ \frac{\partial}{\partial \theta_i}\frac{\partial}{\partial \theta_k} &= -\frac{\partial}{\partial \theta_k}\frac{\partial}{\partial \theta_i}. \end{aligned} \tag{68}$$

One should finally verify that again due to the nilpotency problem differentiation is strictly speaking not well-defined on functions, but only on the corresponding expressions, i.e. in terms of operations on sets of coefficients $a^{(p,q)}_{i_1 \ldots i_p\, j_1 \ldots j_q}$.

For the purpose of this paper we need not only polynomials but more generally C^∞ *functions*. The relevant idea which of course only affects the even variables can be best understood by looking at functions of a single even variable. An arbitrary element x from $\mathfrak{B}^{(0)}_{sc}$ can be written as a unique sum $x = x_b + x_s$, with x_b a real number, as a multiple of the identity element of $\mathfrak{B}$, and x_s nilpotent. (DeWitt [20] calls x_b and x_s body and soul of x). Using this decomposition a given C^∞ function $f:\mathbb{R} \to \mathfrak{B}$ can be extended to a function $\hat{f}$: $\mathfrak{B}^{(0)}_{sc} \to \mathfrak{B}$ by the formula

$$\hat{f}(x) := \sum_{k=0,1,\ldots} \frac{x_s^k}{k!} f^{(k)}(x_b) \tag{69}$$

in which $f^{(k)}$ is the k^{th} derivative of f. The summation involves only a finite number of terms because x_s is nilpotent. The function $\hat{f}$ can be expanded in a not necessarily convergent infinite Taylor series around a fixed value in ν in $\mathfrak{B}^{(0)}_{sc}$, which is up to first order

$$\begin{aligned} \hat{f}(x+\nu) &= \sum_{k=0,1,\ldots} \frac{(x_s+\nu_s)^k}{k!} f^{(k)}(x_b+\nu_b) = \\ &= \sum_{k=0,1,\ldots} \frac{x_s^k + k x_s^{k-1}\nu_s + \ldots}{k!} \left[f^{(k)}(x_b) + \nu_b f^{(k+1)}(x_b) + \ldots \right] = \\ &= \hat{f}(x) + \nu \hat{f}^{(1)}(x) + \ldots \end{aligned} \tag{70}$$

with

$$\hat{f}^{(1)}(x) = \sum_{k=0,1,\ldots} \frac{x_s^k}{k!} f^{(k+1)}(x_b). \tag{71}$$

The derivative of $\hat{f}$ with respect to $x \in \mathfrak{B}^{(0)}_{sc}$ is defined as the first order term $\hat{f}^{(1)}(x)$ in (70). Formula (71) shows that it is in fact the extension of $f^{(1)}(x_b)$, the ordinary derivative of f. This enables us to define in a consistent way C^∞ functions of a variables in $\mathfrak{B}^{(0)}_{sc}$ as functions which can be obtained from C^∞ functions of a real variable by means of the extension formula (69). The definition of a C^∞ function of m even variables is an obvious generalization. For the general case of a function m even and n odd variables we first rewrite (62) as

$$\begin{aligned} & f(x_1,\ldots,x_m,\, \theta_1,\ldots,\theta_n) = \\ & = \sum_{q=0}^{\infty} \frac{1}{p!} \sum_{j_1,\ldots,j_q=1,2,\ldots,n} \theta_{j_1} \ldots \theta_{j_q}\, h_{j_1 \ldots j_q}(x_1,\ldots,x_m) \end{aligned} \tag{72}$$

with

$$h_{j_1 \dots j}(x_1, \dots, x_m) =$$

$$= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{i_1, \dots, i_p = 1,2,\dots,m} x_{i_1} \dots x_{i_p} a^{(p,q)}_{i_1 \dots i_p\, j_1 \dots j_q}. \tag{73}$$

The function f will be a general C^∞ function if we allow the $h_{j_1 \dots j_q}(x_1, \dots, x_m)$, to be C^∞ functions of the even variables $x_1, \dots, x_m$, instead of polynomials.

Berezin *integration* is used in this paper only for a few remarks in a not completely rigorous context in sections V and VI. We restrict ourselves for this reason here to the barest essentials. The Berezin integral over the odd variable θ_j is an operator in the space of functions $f(x_1, \dots, x_m, \theta_1, \dots, \theta_n)$, denoted as $\int d\theta_j$, which is "graded linear", i.e.

$$\int d\theta_j (f + g) = \int d\theta_j f + \int d\theta_j g \tag{74}$$

$$\int d\theta_j (a^{(s)} f) = (-1)^s a^{(s)} \int d\theta_j f, \quad a^{(s)} \in \mathfrak{B}^{(s)}, \quad s = 0,1$$

and is determined by

$$\int d\theta_j a = 0 \quad a \in \mathfrak{B}$$

$$\int d\theta_j \theta_k = \delta_{jk} 1_{\mathfrak{B}}. \tag{75}$$

The multiple integral $\int d\theta_1 \dots d\theta_q$ is by definition the repeated integral $\int d\theta_1 \int d\theta_2 \dots \int d\theta_q$. Different integrals anticommute, i.e. $\int d\theta_j d\theta_k = -\int d\theta_k d\theta_j$. There is also an integral $\int dx_j$ over an even variable x_j. To give a proper definition is somewhat more tricky. It occurs in the superspace action. We have not really used it in this paper and shall therefore not discuss it here.

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REFERENCES

1 J. WESS, B. ZUMINO (1974). *Phys. Lett.* **49B**, 52.
2 J. WESS, J. BAGGER (1983). *Supersymmetry and Supergravity.* Princeton University Press.
3 P. FAYET, S. FERRARA (1977). *Phys. Reports* **32C**, 249.
4 M.F. SOHNIUS (1985). *Phys. Reports* **128**, 39-204.
5 P.G.O. FREUND (1986). Introduction to supersymmetry. *Cambridge University Press.*
6 P. SALOMONSON, J.W. VAN HOLTEN (1982). *Nucl. Phys.* **B196**, 509.
7 E. WITTEN (1982). *Nucl. Phys.* **B202**, 253.

8 M. BATCHELOR (1980). *Trans. Amer. Math. Soc.* **258**, 257.

9 M. BATCHELOR (1984). Graded manifolds and super manifolds. in: "Mathematical aspects of superspace". H.-J. SEIFERT, C.J.S. CLARKE, A. ROSENBLUM, editors. Reidel.

10 A. ROGERS (1980). *J. Math. Phys.* **21**, 1352.

11\. A. ROGERS (1984). Aspects of the geometric approach to supermanifolds. in: "Mathematical aspects of superspace". H.-J. SEIFERT, C.J.S. CLARKE, A. ROSENBLUM, editors. Reidel.

12 A. ROGERS (1985). J. Math. Phys. **26**, 2749.

13 A. ROGERS (1986). *Commun. Math. Phys.* **105**, 375.

14 A. JADCZYK, K. PILCH (1981). *Commun. Math. Phys.* **78**, 373.

15 J. HOYOS, M. QUIROS, J. RAMIREZ-MITTELBRUNN, F.J. DE URRIES (1984). *J. Math. Phys.* **25**, 833, **25**, 841, **25**, 847.

16 C.P. BOYER, S. GITLER (1984). *Trans. Amer. Math. Soc.* **285**, 241.

17 R.F. PICKEN, K. SUNDERMEYER, (1986). *Commun. Math. Phys.* **102**, 585.

18 J.M. RABIN, L. CRANE, (1985). *Commun. Math. Phys.* **100**, 141, (1985); **102**, 123.

19 M.J. ROTHSTEIN (1986). *Trans. Amer. Math. Soc.* **297**, 159.

20 B. DEWITT (1984). Supermanifolds. *Cambridge University Press.*

21 F.A. BEREZIN, D.A. LEITES (1975). *Soviet Math. Dokl.* **16**, 1218.

22 D. LEITES (1980). *Russian Math. Surveys* **35**, 1.

23 B. KOSTANT (1977). Graded manifolds, graded Lie theory and prequantization. in: "Differential geometric methods in mathematical physics". K. BLEULER, editor. Springer.

24 M. BATCHELOR (1979). *Trans. Amer. Math. Soc.* **253**, 329.

25 M. BATCHELOR (1985). *J. Math. Phys.* **26**, 1578.

26 B. KUPERSHMIDT (1984). *Proc. Nat. Acad. U.S.A.* **81**, 6562.

27 R. CASALBUONI (1967). *Nuovo Cimento* **33A**, 115.

28 A. ROGERS (1981). *J. Math. Phys,* **22**, 939.

29 F.A. BEREZIN, G.I. KAC (1970). *Math. U.S.S.R. Sbornik* **11**, 311.

30 F. LANGOUCHE, T. SCHÜCKER (1985). *Lett. in Math. Phys.* **9**, 139.

31 A. SALAM, J. STRATHDEE (1974). *Nucl. Phys.* **B76**, 477.

32 A. SALAM, J. STRATHDEE (1974). *Phys. Rev.* **D11**, 1521.

33 A. SALAM, J. STRATHDEE (1978). *Fortschritte der Physik* **26**, 57.

34 A. ROGERS (1985). *J. Math. Phys.* **26**, 385.

35 A. ROGERS (1986). *J. Math. Phys.* **27**, 710.

36 J. DELL, L. SMOLIN (1979). *Commun. Math. Phys.* **66**, 197.

37 F.A. BEREZIN (1966). The method of second quantization. *Academic Press.*

Mathematical Aspects of Supersymmetric Theories

II. Algebras and Derivations

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TABLE OF CONTENTS

Abstract

Starting from the results of the discussion of a simple model in the preceding paper we develop an algebraic framework for more general supersymmetric field theories. This leads in particular to a coordinate-free picture of the space of classical superfields and some of its properties, both in the extended and in the minimal approach, i.e. with and without the explicit use of Grassmann variables.

1. Introduction

In Ref. 1 we gave a rigorous and fairly detailed account of a simple model of supersymmetric field theory. This allowed us to focus on some of the typical features of supersymmetry at an elementary level and to set the stage for a more systematic discussion of the mathematical structure of supersymmetric field theory such as we shall begin in this paper.

We treated the model of Ref. 1 — more or less the standard Wess-Zumino model reduced to 1-dimensional space-time — in what we have called the extended formalism. This means the explicit use of anticommuting variables as elements of a certain unspecified but fixed auxiliary Grassmann algebra. (In a wider context this is also called the geometric approach.) In our view, this approach to the method of anticommuting variables is intuitively the most obvious and natural one when starting from applications in theoretical physics. We expect nevertheless that the alternative minimal (or algebraic) approach, in which there are no auxiliary Grassmann algebras, will in the end capture the mathematical essence of such applications in a more economical way.

Our point of departure in Section II is therefore a generalization of the results from Ref. 1 on superfields as Grassmann algebra valued functions. We study the properties of such a general system of superfields as a graded commutative algebra A on which a supersymmetry group $\mathfrak{T}_{sc}$ acts by means of automorphisms and the corresponding Lie algebra T_{sc} by derivations. A Grassmann algebra $\mathfrak{B}$, the auxiliary algebra of the extended approach, plays the rôle of basic ring of scalars in all this. The system contains a subsystem, independent of $\mathfrak{B}$, consisting of a smaller graded commutative algebra $A_{\mathbf{C}}$, with a super Lie algebra $T_{\mathbf{C}}$ acting on it, and with $\mathbf{C}$ as ring of scalars. The small system is typical for the minimal approach. It contains the same information as the large one and in fact generates it. How this happens can be understood in a more coordinate-free manner from Section III, where we give a general procedure for extending systems over $\mathbf{C}$ to systems over an auxiliary Grassmann algebra $\mathfrak{B}$. In Section IV we continue our discussion of the algebra of superfields both in its minimal and its extended version. This algebra is in first instance the graded symmetric tensor algebra over the dual of a translation super Lie algebra or its $\mathfrak{B}$-module extension, but it has to be modified somewhat because superfields in physics are not just polynomials but have a C^{∞} dependence on the even variables. These even variables are moreover real in a certain sense, meaning that a suitable real structure must be specified within the general scheme. The basic object in defining superfields and in formulating supersymmetry is a supertranslation algebra or group, as we demonstrated in our discussion of the 1-dimensional model in Ref. 1. The generalization to larger superalgebras and groups connected with inhomogeneous transformations needed for theories in 4-dimensional space-time is relatively straightforward and is carried out in Section V. We apply these ideas in Section VI to the space of general superfields for 4-dimensional space-time, as a system generated by the super Poincaré algebra. We discuss briefly some of the invariant subspaces or 'multiplets' that play a rôle in the standard Wess-Zumino models. In Section VII we sketch the manner in which our results on classical superfields will be used in setting up the corresponding supersymmetric quantum fields. Section VIII is an appendix. It should be consulted, together with Ref. 1, for unexplained terms. It has the purpose to unify notation and conventions and to make this and the preceding paper relatively self-contained. Most but not all of the material in it can be found in various places in the literature. Ref. 1 contains a general list of references.

II. Algebraic Properties of Superfields

In Ref. 1 we obtained the classical superfields $\phi(\tau,\theta)$ of the 1-dimensional Wess-Zumino model as functions on a (1,2)-dimensional supertranslation group, with supersymmetry transformations coming from the left action of the group on itself. This system of fields and transformations has an obvious algebraic structure: The fields form a graded commutative algebra and the group acts by automorphisms of this algebra. In this the basic ring of scalars is a fixed but unspecified Grassmann algebra $\mathfrak{B}$, indicating that we are in what we have called the extended formulation of supersymmetry. A closer inspection of the system shows however that it contains and in fact is generated by a subsystem over $\mathbb{C}$ which is independent of $\mathfrak{B}$ and which describes the situation in the minimal picture. We shall discuss this now in some detail, but it is worthwhile to do it in a more general setting. Supersymmetry is typically and in first instance based on a group of supertranslations. This is obviously the case for the 1-dimensional Wess-Zumino model, which in its quantized version is an example of supersymmetric quantum mechanics. It holds for $N=1$ supersymmetry in ordinary space-time and after a suitable central extension for extended supersymmetry. In supergravity one has in the spirit of gauge theory local supertranslation groups in every point of super space-time.

In the following $\mathfrak{B}=\mathfrak{B}^{(0)}\oplus\mathfrak{B}^{(1)}$ will be a complex Grassmann algebra of fixed but unspecified finite dimension. We use the notation $\mathfrak{B}^p$ for the p-fold Cartesian product $\times^p\mathfrak{B}$ and $\mathfrak{B}^{m,n}$ for $(\times^m\mathfrak{B}^{(0)})\times(\times^n\mathfrak{B}^{(1)})$. $\mathfrak{B}$ has a conjugation $b\to b^*$ and we denote the 'real', i.e. self-conjugate part of $\mathfrak{B}$ as $\mathfrak{B}_{sc}$. Similarly we have $\mathfrak{B}_{sc}^{(0)}$, $\mathfrak{B}_{sc}^{m,n}$, etc.

Suppose to be given a system of m symmetric $n\times n$ matrices s^j, $j=1,\ldots,m$, with real-valued matrix elements s^j_{kl}. We then define the (m,n)-*dimensional supertranslation group* $\mathfrak{T}_{sc}$ as the set $\mathfrak{B}_{sc}^{m,n}$ provided with the group multiplication

$$(a,\epsilon)(a',\epsilon') = (a+a'+i\sum_{l,k=1}^{n} s_{kl}\epsilon^k\epsilon'^l,\epsilon+\epsilon'). \tag{1}$$

In this the elements of $\mathfrak{B}_{sc}^{m,n}$ are written as pairs from $\mathfrak{B}_{sc}^{m,0}\times\mathfrak{B}_{sc}^{0,n}$. We shall write indices as much as possible in upper and lower positions in such a way that we can use the Einstein summation convention. This means that we will omit from now on summation signs as in (1). $\mathfrak{T}_{sc}$ is in any case a real Lie group of dimension $2^{(m+n)(N-1)}$, with 2^N the dimension of $\mathfrak{B}$, but in this context it is more fitting to look at it as a (m,n)-dimensional super Lie group. A precise definition of this concept has been given by Rogers[2] but because of the explicitness and simplicity of the situation there is no need to invoke this here. Note that the elements $(0,\epsilon)$, which may be called proper supertranslations, do not form a subgroup of $\mathfrak{T}_{sc}$. One should finally observe that the special case of the superfields of our 1-dimensional Wess-Zumino model is recovered from this general set up by taking $m=1$, $n=2$, with the single 2×2 matrix s_{lk} equal to the unit matrix δ_{lk}. Let A be the collection of functions ϕ: $\mathfrak{B}_{sc}\rightarrow\mathfrak{B}$ which are C^∞ in the sense given in the appendix of Ref. 1, meaning that they can be written as

$$\begin{aligned}\phi(x,\theta) &= \phi(x^1,\ldots,x^m,\theta^1,\ldots,\theta^n) \\ &= \sum_{q=0,1,\ldots,n} \frac{1}{q!}\theta^{j_1}\cdots\theta^{j_q}\phi_{j_1\cdots j_q}(x)\end{aligned} \tag{2}$$

with the $\phi_{j_1\cdots j_q}$, antisymmetric in $j_1\cdots j_q$, C^∞ functions from $\mathfrak{B}_{sc}^{m,0}$ to $\mathfrak{B}$ and as such extensions of C^∞ functions from $\mathbb{R}^m$ to $\mathfrak{B}$ according to

$$f(x) = f(x_S+x_B) = \sum_{k=0} \frac{1}{k!} x_S^{i_1}\cdots x_S^{i_k}(\delta_{i_1}\cdots\delta_{i_k}f)(x_B) \tag{3}$$

A has a grading: ϕ is homogeneous of degree α if the degree $|\phi(x,\theta)|=\alpha$, $\forall(x,\theta)\in\mathfrak{B}_{sc}^{m,n}$. This is consistent with the pointwise multiplication of functions; $(\phi_1\phi_2)(x,\theta)=\phi_1(x,\theta)\phi_2(x,\theta)$, and with the conjugation of functions defined by using the conjugation in $\mathfrak{B}$ as $\phi^*(x,\theta)=(\phi(x,\theta))^*$. With all this A is a *graded commutative algebra over* $\mathfrak{B}$, with conjugation. The functions ϕ in A are the $\mathfrak{B}$-*valued superfields* in this setting. The $\phi_{j_1\cdots j_q}$ are what is usually called the *component fields.*

We next define the *left regular representation* of $\mathfrak{T}_{sc}$ in A as

$$(W(a,\epsilon)\phi)(x,\theta) = \phi((a,\epsilon)^{-1}(x,\theta)) = \phi(x-a-is_{kl}\epsilon^k\theta^l,\theta-\epsilon) \tag{4}$$

The $W(a,\epsilon)$ are even linear operators in A and in fact *automorphisms* of A as a graded algebra. $A^{(0)}$ is invariant and also $A_{sc}^{(0)}$ because the $W(a,\epsilon)$ commute with the conjugation in A.

Differentiation of 1-parameter subgroups $W(ta,t\epsilon)$ with respect to $t\in\mathbb{R}$ gives a representation of the Lie algebra $T_{sc}^{(0)}$ by means of differential operators

$$E(a,\epsilon) = a^jE_j^{(0)}+\epsilon^kE_k^{(1)} \tag{5}$$

with

$$\begin{aligned}E_j^{(0)} &= -\frac{\partial}{\partial x^j} \\ E_k^{(1)} &= -\frac{\partial}{\partial\theta^k}-is_{kl}^j\theta^l\frac{\partial}{\partial x^j}\end{aligned} \tag{6}$$

The $E(a,\epsilon)$ are again even $\mathfrak{B}$-linear operators in A, leaving $A^{(0)}$ and $A_{sc}^{(0)}$ invariant and moreover even *derivations* of A. $T_{sc}^{(0)}$ is a Lie algebra over $\mathbb{R}$ but also over $\mathfrak{B}_{sc}^{(0)}$. Its elements can also be written as pairs $(a,\epsilon)\in\mathfrak{B}_{sc}^{m,n}$ with Lie bracket

$$[(a,\epsilon),(a',\epsilon')] = (2is_{kl}\epsilon^k\epsilon'^l,0) \quad \forall (a,\epsilon),(a',\epsilon')\in\mathfrak{B}_{sc}^{m,n} \tag{7}$$

The differential operators $E(a,\epsilon)$ are well-defined not only for $(a,\epsilon)\in\mathfrak{B}_{sc}^{m,n}$ but more generally for $(a,\epsilon)\in\mathfrak{B}^{m+n}$. The general operators $E(a,\epsilon)$ are derivations of degree α if $\alpha=|a^1|=\cdots=|a^m|=|\epsilon^1|+1=\cdots=|\epsilon^n|+1$, form a closed system under graded commutation and represent a *super Lie algebra* T over $\mathfrak{B}$, consisting of elements $(a,\epsilon)\in\mathfrak{B}^{m+n}$ with graded Lie bracket

$$[(a,\epsilon),(a',\epsilon')] = (2i(-1)^{|(a',\epsilon')|}s_{kl}\epsilon^k\epsilon'^l,0) \tag{8}$$

for all (a,ϵ) and (a',ϵ') from $\mathfrak{B}^{m+n}$ and with the degree $|(a',\epsilon')|=|a'^1|=\cdots=|a'^m|=|\epsilon'^1|+1=\cdots=|\epsilon'^n|+1$. There is a conjugation in T given by

$$(a,\epsilon)^* = (a^*,(-1)^{|(a,\epsilon)|}\epsilon^*) \tag{9}$$

It is determined by the requirement that the initial Lie algebra $T_{sc}^{(0)}$ is indeed the even self-conjugate part of T and that the extension of the representation E from $T_{sc}^{(0)}$ to T is self-conjugate. It is helpful to see the (a,ϵ) as left coordinates. T is then a $\mathfrak{B}$-module with basis $e_1^{(0)},\ldots,e_m^{(0)},e_1^{(1)},\ldots,e_m^{(1)}$ and an element $u\in T$ is written as $u=a^je_j^{(0)}+\epsilon^ke_k^{(1)}$. Definition (9) is equivalent to $e_j^{(0)*}=e_j^{(0)}$ and $e_k^{(1)*}=-e_k^{(1)}$ and instead of (8) we have the simple graded Lie brackets

$$\begin{aligned}&[e_j^{(0)},e_k^{(0)}]=0 \quad [e_j^{(0)},e_k^{(1)}]=0\\ &[e_k^{(1)},e_l^{(1)}] = -2is_{kl}^je_j^{(0)}\end{aligned} \tag{10}$$

For the operators $E_j^{(\alpha)}=E(e_j^{(\alpha)})$ one verifies that the following properties hold

$$\begin{aligned}(E_j^{(0)}\phi)^* &= E_j^{(0)}\phi^*\\ (E_k^{(0)}\phi)^* &= -(-1)^{|\phi|}E_k^{(1)}\phi^* \qquad \forall\phi\in A\end{aligned} \tag{11}$$

This implies the self-conjugacy property for general $E(m)$ expressed by formula (84) from the appendix.

Together with the $W(a,\epsilon)$ there is a *right regular representation* in A defined as

$$(W_R(a,\epsilon)\phi)(x,\theta) = \phi((x,\theta)(a,\epsilon)) = \phi(x+a-is_{kl}\epsilon^k\epsilon^l,\theta+\epsilon) \tag{12}$$

It commutes with the left regular representation and has similar properties. Its usefulness will become clear in Section VI.

Summing up this section so far we see that the superfields in the extended picture form a graded commutative algebra A, over $\mathfrak{B}$ and with conjugation. The supertranslation group $\mathfrak{T}_{sc}$ is represented by automorphisms of A, its Lie algebra $T_{sc}^{(0)}$ by derivations. The representation of $T_{sc}^{(0)}$ has a natural extension to a representation of a super Lie algebra T with conjugation, such that $T_{sc}^{(0)}$ is the even self-conjugate part of T and the representation of T is self-conjugate.

We return finally to the subsystem over $\mathbb{C}$ which is contained in this system over $\mathfrak{B}$ and describes the situation from the minimal point of view. Let $A_{\mathbb{C}}\subset A$

consist of all ϕ such that the component functions $\phi_{j_1 \ldots j_q}(x)$ in (2) take values in $\mathbb{C} \subset \mathfrak{B}$ when restricted to $x \in \mathbb{R}^m \subset \mathfrak{B}^{m,0}$. $A_{\mathbb{C}}$ is a graded commutative algebra over $\mathbb{C}$ and can clearly be defined independently of $\mathfrak{B}$. It is crucial for the understanding of the relation between the extended and the minimal formulation of supersymmetry to observe that $A_{\mathbb{C}}$ is *not* invariant under the action of the supertranslation group $\mathfrak{T}_{sc}$ as represented by the operators $W(a, \epsilon)$. This means that the supergroup $\mathfrak{T}_{sc}$ does not act in $A_{\mathbb{C}}$ as a separate algebra and neither does its Lie algebra $T_{sc}^{(0)}$ or the larger super Lie algebra T. The latter however has a natural complex super Lie subalgebra which is compatible with $A_{\mathbb{C}}$. We call it $T_{\mathbb{C}}$ and it consists of all $u = a^j e_j^{(0)} + \epsilon^k e_k^{(1)}$ with coordinates (a, ϵ) in $\mathbb{C}^{m+n}$. This mean that it is spanned by the same basis $e_1^{(0)}, \ldots, e_m^{(0)}, e_1^{(1)}, \ldots, e_n^{(1)}$ with the corresponding graded Lie bracket of (10). One verifies immediately that the representation E restricted to this $T_{\mathbb{C}}$ leaves $A_{\mathbb{C}}$ invariant. The subsystem consisting of $T_{\mathbb{C}}$ and its representation by derivations of $A_{\mathbb{C}}$ contains the essential ingredients of the situation in an adequate and very economical manner even though one has only infinitesimal supertransformations. It seems therefore natural to reverse the direction of the procedure of this section and start with the simpler complex system when setting up general supersymmetric theories. From this an extended formulation with scalars from an auxiliary Grassmann algebra $\mathfrak{B}$ can be obtained when needed. The general scheme for doing this in a coordinate-free manner will be given in the next section.

III. An extension procedure

Consider to be given a system (L,R,π) consisting of a super Lie algebra L, a graded commutative algebra R, both over $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, and a representation π of L by derivations of R.

Choose a fixed Grassmann algebra $\mathfrak{B}$, also over $\mathbb{F}$ and of finite dimension. We extend (L,R,π) to a system $(L^{\mathfrak{B}}, R^{\mathfrak{B}}, \pi^{\mathfrak{B}})$, in which $\mathfrak{B}$ has replaced $\mathbb{F}$ as the basic ring of scalars, by the following steps:

1. Define $L^{\mathfrak{B}} = \mathfrak{B} \otimes_{\mathbb{F}} L$. $L^{\mathfrak{B}}$ is a $\mathfrak{B}$-module with $b_1(b_2 \otimes_{\mathbb{F}} u) = b_1 b_2 \otimes_{\mathbb{F}} u$; $\forall b_1, b_2 \in \mathfrak{B}$, $u \in L$, and becomes a super Lie algebra over $\mathfrak{B}$ by introducing a graded Lie bracket as

$$[b_1 \otimes_{\mathbb{F}} u_1, b_2 \otimes_{\mathbb{F}} u_2] = (-1)^{|b_1||b_2|} b_1 b_2 \otimes_{\mathbb{F}} [u_1, u_2] \quad (13)$$

$$\forall b_1, b_2 \in \mathfrak{B},\ u_1, u_2 \in L$$

2. Define $R^{\mathfrak{B}} = \mathfrak{B} \otimes_{\mathbb{F}} R$. $R^{\mathfrak{B}}$ is a $\mathfrak{B}$-module and a graded commutative algebra over $\mathfrak{B}$ with multiplication given by

$$(b_1 \otimes_{\mathbb{F}} \phi_1)(b_2 \otimes_{\mathbb{F}} \phi_2) = (-1)^{|\phi_1||b_2|} b_1 b_2 \otimes_{\mathbb{F}} \phi_1 \phi_2 \quad (14)$$

$$\forall b_1, b_2 \in \mathfrak{B},\ \phi_1, \phi_2 \in R$$

3. Finally we extend the representation π to a representation $\pi^{\mathfrak{B}}$ of $L^{\mathfrak{B}}$ in $R^{\mathfrak{B}}$ by

$$\pi^{\mathcal{B}}(b_1 \otimes_{\mathbb{F}} u)(b_2 \otimes_{\mathbb{F}} \phi) = (-1)^{|u||b_2|} b_1 b_2 \otimes_{\mathbb{F}} \pi(u)\phi \qquad (15)$$

$$\forall b_1, b_2 \in \mathcal{B},\ u \in L,\ \phi \in R$$

One checks easily that $\pi^{\mathcal{B}}(b_1 \otimes_{\mathbb{F}} u)$ is a left $\mathcal{B}$-linear operator in $R^{\mathcal{B}}$ as a $\mathcal{B}$-module, of degree $|b_1 \otimes_{\mathbb{F}} u| = |b_1| + |u|$, and is moreover a derivation of $R^{\mathcal{B}}$ as an algebra. We can of course identify L and R with subsets of $L^{\mathcal{B}}$ and $R^{\mathcal{B}}$ and may write according bu and $b\phi$ instead of $b \otimes_{\mathbb{F}} u$ and $b \otimes_{\mathbb{F}} \phi$.

Note that the system $(L^{\mathcal{B}}, R^{\mathcal{B}}, \pi^{\mathcal{B}})$ can also be obtained by tensoring from the right with $\mathcal{B}$, with appropriate modifications in the formulas (13), (14) and (15). The resulting system is canonically isomorphic and will not be distinguished.

The extension procedure can be put in a somewhat more general setting by starting with a system (L, R, W, π) with W an R-module and π a representation of L by derivations of R and at the same time by derivations of W in the sense that $\pi(u)\phi\psi = (\pi(u)\phi)\psi + (-1)^{|u||\phi|}\phi(\pi(u)\psi)$, $\forall u \in L$, $\phi \in R$, $\psi \in W$.The result of tensoring with $\mathcal{B}$ will be system $(L^{\mathcal{B}}, R^{\mathcal{B}}, W^{\mathcal{B}}, \pi^{\mathcal{B}})$ with corresponding properties but with $\mathcal{B}$ instead of $\mathbb{F}$ as ring of scalars. Except for brief remarks in Sections V and VI this generalization will not be needed in the present discussion.

The general merit of this extension procedure is that it gives the possibility of associating proper group actions with super Lie algebras and their representations. The even part $(L^{\mathcal{B}})^{(0)}$ of $L^{\mathcal{B}}$ is an ordinary Lie algebra over $\mathcal{B}^{(0)}$ (and of course over $\mathbb{F}$) and can as such be related to a *supergroup* in some precise sense. The restriction of the representation $\pi^{\mathcal{B}}$ to this even part of $L^{\mathcal{B}}$ consists of even derivations which can in principle be exponentiated to give a representation of the supergroup in terms of automorphisms of $R^{\mathcal{B}}$.

We specialize in this paper to the case where the initial system is over $\mathbb{F} = \mathbb{C}$ and has an additional real structure, which means that L and R have conjugations such that the representation is self-conjugate. This ensures that appropriate self-conjugacy properties reappear in the extended system after conjugations in $L^{\mathcal{B}}$ and $R^{\mathcal{B}}$ have been defined according to

$$(b \otimes_{\mathbb{F}} w)^* = (-1)^{|b||w|} b^* \otimes_{\mathbb{C}} w^* \qquad \forall b \in \mathcal{B},\ w \in L \text{ or } R \qquad (16)$$

In our application R will be moreover such that only the representation of the self-conjugate part $(L^{\mathcal{B}})^{(0)}_{sc}$ of $(L^{\mathcal{B}})^{(0)}$ can be exponentiated in $R^{\mathcal{B}}$.

The idea of going from $\mathbb{C}$ to a Grassmann algebra $\mathcal{B}$ by tensoring with $\mathcal{B}$ operates in different forms and at different places in the mathematical formulation of supersymmetric theories. The particular extension procedure given in this section underlies the notion of classical superfields in its various guises, as will be clear in the next sections. A similar construction plays a rôle on the level of quantum theory where $\mathcal{H}$, the Hilbert space of state vectors, can be extended to $\mathcal{H}^{\mathcal{B}} = \mathcal{B} \otimes_{\mathbb{C}} \mathcal{H}$.

IV. A MINIMAL AND AN EXTENDED APPROACH TO SUPERFIELDS. THE ALGEBRAS $S_{sc}(M')$ AND $S_{sc}^{\infty}(M')$.

The polynomials on a finite dimensional vector space M form a commutative algebra $P(M)$, which is in a natural manner isomorphic to $S(M')$, the algebra of symmetric tensors over the dual M'. (In general a polynomial map ϕ from a vector space V to a vector space W is a map that can be written as the finite sum of diagonally evaluated symmetric multilinear maps $\phi^{(n)}$: $V\times\cdots\times V\to W$, $n=0,1,2,\ldots$. The space of such maps is denoted as $P(V;W)$, with the special care $P(V;\mathbb{F})=P(V)$.) Under the isomorphism $P(M)\cong S(M')$ operations on polynomials in $P(M)$ such as partial differentiation and multiplication correspond to the algebraic operations in $S(M')$ that appear in the appendix as annihilation and creation operators.

For a graded vector space M with dual M' one has the *graded symmetric tensor algebra* $S(M')$. The basic idea underlying the heuristic method of anticommuting variables amounts to pretending that this $S(M')$ is isomorphic to an algebra of 'polynomial functions of commuting and anticommuting variables'. Annihilation and creation operators in $S(M')$ are interpreted as differentiation and multiplication operations on these 'functions'.

There are two ways to work in a precise manner with this idea. In the *minimal* (or algebraic) formalism a 'polynomial of commuting and anticommuting variables' is not a function but just an element of the algebra $S(M')$. In the *extended* (or geometric) picture one extends $S(M')$ to $S(M')^{\mathfrak{B}}$ in the manner described in Section III. A 'polynomial' is then an element of this extended algebra but can now also be regarded as a proper function, a polynomial function on the even part of the module M with values in $\mathfrak{B}$.

In standard supersymmetric field theory a classical (scalar) superfield is a 'function' — in one of the two precise meanings indicated above — on a graded vector space M, consisting, as we shall discuss in more detail in Section VI, of 4-dimensional complexified Minkowski space-time as even part $M^{(0)}$ and an equally 4-dimensional spinor space as $M^{(1)}$. Fields in this context should however not be polynomials but C^{∞} functions, which of course means something extra for the even variables only. This C^{∞} dependence should moreover be in terms of real variables, i.e. from the self-conjugate part $M_{sc}^{(0)}$. In order to meet these requirements we introduce an algebra which is larger than $S(M')$ and which will be denoted as $S_{sc}^{\infty}(M')$:

Let M be a finite dimensional graded vector space over $\mathbb{C}$, with conjugation. Forgetting the grading we have the linear isomorphisms $S(M')\cong S(M'^{(0)})\otimes\Lambda(M'^{(1)})\cong P(M^{(0)})\otimes\Lambda(M'^{(1)})\cong P(M^{(0)};\ \Lambda(M'^{(1)}))$ and by restricting the polynomial maps the final isomorphism (still over $\mathbb{C}$!) $P(M^{(0)};\ \Lambda(M'^{(1)}))\cong P(M_{sc}^{(0)};\ \Lambda(M'^{(1)}))$. This means that $S(V')$ has a realization as an algebra of polynomial maps, not from M to $\mathbb{C}$, but from $M_{sc}^{(0)}$ to $\Lambda(M'^{(1)})$. As such it can be embedded in the larger algebra $C^{\infty}(M_{sc}^{(0)};\ \Lambda(M'^{(1)}))$

of C^∞ function from $M_{sc}^{(0)}$ to $\Lambda(M'^{(1)})$. The *even* annihilation and creation operators in $S(M')$ appear in $P(M_{sc}^{(0)};\ \Lambda(M'^{(1)}))$ as ordinary differentiation and multiplication operators. In this form they extend immediately to $C^\infty(M_{sc}^{(0)};\ \Lambda(M'^{(1)}))$. The odd operators are algebraic operations on the factor $\Lambda(M'^{(1)})$ in $S(M') \cong P(M_{sc}^{(0)} \otimes_{\mathbb{R}} \Lambda(M'^{(1)})$ and remain so in the extension to $C^\infty(M_{sc}^{(0)};\ \Lambda(M'^{(1)})) \cong Cs(M_{sc}^{(0)} \otimes_{\mathbb{R}} \Lambda(M'^{(1)})$. We look at $C^\infty(M_{sc}^{(0)};\ \Lambda(M'^{(1)}))$ as an extension of $S(M')$ and call it therefore $S_{sc}^{\infty}(M')$. It possesses in an obvious way the structure of a graded commutative algebra over $\mathbb{C}$, with conjugation, a structure which is compatible with that of $S(M')$. It has annihilation operators $A(u)$, $\forall u \in M$, and creation operators $C(v)$, $\forall v \in M'$, which extend those in $S(M')$ and retain there algebraic properties such as expressed e.g. in (58) and (61). (The 'one-particle operators' $\sigma(T)$ also have natural extensions.)

It may be helpful to write all this in an explicit basis dependent form. Let M be (m,n)-dimensional and choose a basis $e_1^{(0)}, \ldots, e_m^{(0)}, e_1^{(1)}, \ldots, e_n^{(1)}$ with $(e_j^{(0)})^* = e_j^{(0)}$ and $(e_j^{(1)})^* = -e_j^{(1)}$. In M' there is the dual basis $e^{(0)1}, \ldots, e^{(0)m}, e^{(1)1}, \ldots, e^{(1)n}$ determined by $\langle e_j^{(\alpha)};\ e^{(\beta)k} \rangle = \delta_{\alpha\beta}\delta_j^k$ and with $(e^{(0)j})^* = e^{(0)j}$, $(e^{(1)j})^* = e^{(1)j}$, because of formula (82). An arbitrary element $\phi \in S(M')$ can be written as

$$\phi = \sum_{\substack{p=0,1,\cdots \\ q=0,1,\ldots,n}} \frac{1}{p!}\,\frac{1}{q!} e^{(0)i_1} \ldots . e^{(0)i_p} e^{(1)j_1} \ldots . e^{(1)j_q} \alpha^{(p,q)}_{i_1 \cdots i_p;\, j_1 \cdots j_q} \tag{17}$$

Compare this with formula (63) in the appendix. We have interchanged upper and lower positions of indices because the M in this and the following sections corresponds to V' there.

The summation in (17) can be carried out in two steps

$$\phi^{(q)}_{j_1 \cdots j_q} = \sum_{p=0,1,\ldots} \frac{1}{p!} e^{(0)i_1} \cdots e^{(0)i_p} \alpha^{(p,q)}_{i_1 \cdots i_p;\, j_1 \cdots j_q} \tag{18}$$

$$\phi = \sum_{q=0,1,\ldots,n} \frac{1}{q!} e^{(1)j_1} \cdots e^{(1)j_q} \phi^{(q)}_{j_1 \cdots j_q} \tag{19}$$

The $\phi^{(q)}_{j_1 \cdots j_q}$ can be interpreted as polynomials from $M_{sc}^{(0)}$ to $\mathbb{C}$, or by writting $u^{(0)} \in M_{sc}^{(0)}$ as $u^{(0)} = x^i e_i^{(0)}$, $x^i \in \mathbb{R}$, from $\mathbb{R}^m$ to $\mathbb{C}$, according to

$$\phi^{(q)}_{j_1 \cdots j_q}(x^1, \ldots, x^m) = \sum_{q=0,1,\cdots} \frac{1}{p!} x^{i_1} \cdots x^{i_p} \alpha^{(p,q)}_{i_1 \cdots i_p;\, j_1 \cdots j_q} \tag{20}$$

Together they add up to a single function from $M_{sc}^{(0)}$ to $\Lambda(M'^{(1)})$ or from $\mathbb{R}^m$ to $\Lambda(M'^{(1)})$

$$\phi(x^1, \ldots, x^m) =$$

$$= \sum_{q=0,1,\ldots,n} \frac{1}{q!} e^{(1)j_1} \cdots e^{(1)j_q} \phi^{(q)}_{j_1 \cdots j_q}(x^1, \ldots, x^m) \tag{21}$$

Note that the basis dependent annihilation and creation operators of even degree become indeed differentiation and multiplication operators in the ordinary sense

$$A_i^{(0)} = \frac{\partial}{\partial x^i} \qquad C^{(0)i} = x^i \tag{22}$$

We may and will use a similar notation for the odd operators although this has only a symbolic meaning here

$$A_j^{(1)} = \frac{\partial}{\partial \theta^j} \qquad C^{(1)j} = \theta^j \tag{23}$$

In this basis dependent formulation the extension from $S(M')$ to $S_{sc}^{\infty}(M')$ can be effected simply by allowing the $\phi_{j_1 \cdots j_q}^{(q)}$ in (19) to be C^{∞} functions instead of just polynomial functions.

The result of the foregoing is that a *superfield on M* in the *minimal* or *algebraic picture* is an element of $S_{sc}^{\infty}(M')$ and as such a C^{∞} function from $M_{sc}^{(0)}$ to $\Lambda(M'^{(1)})$. The explicit functions $\phi_{j_1 \cdots j_q}(x^1, \ldots, x^m)$ are the *component fileds* from the physics literature.

The vectors $e^{(0)i_1} \cdots e^{(0)i_p} e^{(1)j_1} \cdots e^{(1)j_q}$ that span $S(M')$ can also be used for a (left) basis for $S(M')^{\mathfrak{B}}$, the algebra of polynomials in the extended picture. This means that an arbitrary element ϕ from $S(M')^{\mathfrak{B}}$ is represented by (17) with the coefficients $\alpha_{i_1 \cdots i_p; j_1 \cdots j_q}^{(p;q)}$ taking values in $\mathfrak{B}$ instead of $\mathbb{C}$. It is obvious that such a ϕ gives a proper polynomial function from $\mathfrak{B}_{sc}^{m,n}$ to $\mathfrak{B}$ according to

$$\begin{aligned} &\phi(x^1, \ldots, x^m, \theta^1, \ldots, \theta^m) = \\ &= \sum_{\substack{p=0,1,\cdots \\ q=0,1,\ldots,n}} \frac{1}{p!}\frac{1}{q!} x^{i_1} \cdots x^{i_m} \theta^{j_1} \cdots \theta^{j_n} \alpha_{i_1 \cdots i_p; j_1 \cdots j_q}^{(p;q)} \\ &\forall x^i \in \mathfrak{B}_{sc}^{(0)},\ \theta^j \in \mathfrak{B}_{sc}^{(1)} \end{aligned} \tag{24}$$

For an explicit description of elements of $S_{sc}^{\infty}(M')$, the C^{∞} algebra of the extended picture, one turns to formula (19). In Section II and in the appendix of Ref. 1 we discussed the definition of C^{∞} functions from $\mathfrak{B}_{sc}^{m,0}$ to $\mathfrak{B}$ as extensions of ordinary C^{∞} functions from $\mathbb{R}^m$ to $\mathfrak{B}$. If we allow such C^{∞} function $\phi_{j_1 \cdots j_q}^{(q)}$ in (19) we obtain the elements of $S_{sc}^{\infty}(M')^{\mathfrak{B}}$. They give functions from $\mathfrak{B}_{sc}^{m,n}$ (or $M^{\mathfrak{B}})^{(0)}$) to $\mathfrak{B}$ according to

$$\begin{aligned} &\phi(x^1, \ldots, x^m, \theta^1, \ldots, \theta^n) = \\ &= \sum_{q=0,1,\ldots,n} \frac{1}{q!} \theta^{j_1} \cdots \theta^{j_q} \phi_{j_1 \cdots j_q}^{(q)}(x^1, \ldots, x^n) \\ &\forall x^i \in \mathfrak{B}_{sc}^{(0)},\ \theta^j \in \mathfrak{B}_{sc}^{(1)} \end{aligned} \tag{25}$$

The results of the $\mathbb{C} \rightarrow \mathfrak{B}$ extension procedure is that a *superfield on M* or rather *on $(M)^{(0)}$* in the *extended* or *geometric* picture is an element of $S_{sc}^{\infty}(M')^{\mathfrak{B}}$ and gives a such a $\mathfrak{B}$-valued C^{∞} function on the even self-conjugate part $(M^{\mathfrak{B}})^{(0)}$

of the module $M^{\mathfrak{B}}$. The annihilation and creation operators, both even and odd, as expressed in (22) and (23) have in this picture a proper meaning as differentiation operators in the sense of Berezin and of multiplication operators.

We would finally note that the algebras $S^{\infty}_{sc}(M')^{\mathfrak{B}}$ and $S^{\infty}_{sc}(M')$ (as subalgebra $S^{\infty}_{sc}(M')\otimes 1_{\mathfrak{B}} \subset S^{\infty}_{sc}(M')^{\mathfrak{B}})$ give algebras of functions ϕ: $(M^{\mathfrak{B}})^{(0)}_{sc} \to \mathfrak{B}$ which are closely related to the G^{∞} and H^{∞} of Rogers[3]. The context there is purely real, while ours is complex with a real substructure.

V. General supertransformations

Let $M = M^{(0)} \oplus M^{(1)}$ be a complex (m,n)-dimensional graded vector space with conjugation. Choose a symmetric bilinear map $s: M^{(1)} \times M^{(1)} \to M^{(0)}$, which is real in the sense that $s(u_1^{(1)}, u_2^{(1)})^* = s(u_1^{(1)*}, u_2^{(1)*})$, $\forall u_1^{(1)}, u_2^{(1)} \in M^{(1)}$. We use s to make M into a super Lie algebra by defining a graded Lie bracket on $M^{(1)} \times M^{(1)}$ as $[u_1^{(1)}, u_2^{(1)}] = -2is(u_1^{(1)}, u_2^{(1)})$ and equal to 0 on $M^{(0)} \times M^{(0)}$ and $M^{(0)} \times M^{(1)}$. The factor -2 is a convention which gives agreement later on with results of Section II, the imaginary i insures in combination with the reality condition of s that the conjugation in M is a conjugation of super Lie algebras, i.e. satisfies (83). M provided with $[\cdot,\cdot]$ in this way is a complex (m,n)-dimensional super Lie algebra with conjugation and may be called an *algebra of supertranslations.*

We construct two representations of M in $S^{\infty}_{sc}(M')$, the algebra of C^{∞}-superfields on M in the minimal picture, as defined in Section IV. One has ad and $\tilde{ad}$ as the adjoint and co-adjoint representations of M, in M and M' respectively. This means elements $\mathrm{ad}\, u \in \mathfrak{L}(M)$ defined as $(ad\, u)u_1 = [u, u_1]$ and $\tilde{ad}\, u \in \mathfrak{L}(M')$ with $\langle u_1; (\tilde{ad}\, u)v\rangle = -(-1)^{|u||u_1|}\langle (ad\, u)u_1; v\rangle$ $\forall u, u_1 \in M, v \in M'$. We use the annihilation operators $A(\cdot)$ and the 'one-particle operators' $\sigma(\cdot)$ and define operators $E_L(u)$ in $S^{\infty}_{sc}(M')$ as

$$E_L(u) = -A(u) + \frac{1}{2}\sigma(\tilde{ad}\, u) \qquad \forall u \in M \tag{26}$$

Note that these $E_L(u)$ would be well-defined operators for an arbitrary (m,n)-dimensional super Lie algebra M but would in general not form a representation of M. Sufficient for this is however that M is nilpotent in the strong sense that $[u_1, [u_2, u_3]] = 0$, $\forall u_1$, u_2, $u_3 \in M$, which is the case here. It implies $ad[u_1, u_2] = 0$ and therefore also $\tilde{ad}[u_1, u_2] = 0$, $\forall u_1$, $u_2 \in M$. Using this, (60) and (61) one obtains on one hand

$$\begin{aligned} &[E_L(u_1), E_L(u_2)] = \\ &= -\frac{1}{2}[A(u_1), \sigma(\tilde{ad}\, u_2)] - \frac{1}{2}[\sigma(\tilde{ad}\, u_1), A(u_2)] + \frac{1}{4}[\sigma(\tilde{ad}\, u_1), \sigma(\tilde{ad}\, u_2)] = \\ &= -A([u_1, u_2]) + \frac{1}{4}\sigma(\tilde{ad}[u_1, u_2]) = -A([u_1, u_2]) \end{aligned} \tag{27}$$

and on the other hand

$$E_L([u_1, u_2]) = -A([u_1, u_2]) + \frac{1}{2}\sigma(\tilde{ad}[u_1, u_2]) = -A([u_1, u_2])$$

$$\forall u_1, u_2 \in M \tag{28}$$

one checks that the $E_L(u)$ are derivations of degree $|u|$, and have the self conjugate property (84).

There is a second representation E_R defined as

$$E_R(u) = A(u) + \frac{1}{2}\sigma(\tilde{ad}\, u) \qquad \forall u \in M \tag{29}$$

We call E_L and E_R the *left* and *right regular representations* of the supertranslation algebra M, for reasons that will become clear. The two representations are related by a vanishing graded commutator

$$[E_L(u_1), E_R(u_2)] = 0 \qquad \forall u_1, u_2 \in M \tag{30}$$

The left regular representation E_L, to be denoted as E from now on, will describe the supersymmetry transformations of fields. E_R will give *constraints* which reduce the space of these fields.

The super Lie algebra M and its representation E (or E_R) form a system $(M, S^{\infty}_{sc}(M'), E)$ on which the extension scheme from Section III can be applied. The result is a system $(M^{\mathfrak{B}}, S^{\infty}_{sc}(M')^{\mathfrak{B}}, E^{\mathfrak{B}})$. The restriction of E^B to $(M^{\mathfrak{B}})^{(0)}$ can be exponentiated to a representation of a supergroup which is in fact $(M^{\mathfrak{B}})^{(0)}_{sc}$ again but now provided with multiplication

$$u_1 \cdot u_2 = u_1 + u_2 - \frac{1}{2}[u_1, u_2] \qquad \forall u_1, u_2 \in (M^{\mathfrak{B}})^{(0)}_{sc} \tag{31}$$

Note that this multiplication is not distributive and that the associativity is assured only by the strong nilpotency property of $M^{\mathfrak{B}}$ as a super Lie algebra. Choosing a bases $e^{(0)}_1,..,e^{0}_m, e^{(1)}_1, \ldots, e^{(0)}_n$ in M (and consequently in $M^{\mathfrak{B}}$) with the property $e^{(0)*}_j = e^{(0)}_j, e^{(1)*}_k = -e^{(1)}_k$ leads to the identification of this group with the supertranslation group $\mathfrak{T}_{sc}$ from Section II. One has furthermore $M^{\mathfrak{B}} = T$, $M = T_{\mathbb{C}}$, $S^{\infty}_{sc}(M')^{\mathfrak{B}} = A$, $S^{\infty}_{sc}(M') = A_{\mathbb{C}}$ and one recovers all the explicit formulas and results there.

Supersymmetry for the 1-dimensional model that we took as our point of departure in Ref. 1 is just symmetry with respect to a supertranslation group or algebra. In supersymmetric field theory symmetry group is larger and contains a group of homogeneous space-time transformations such as the homogeneous Lorentz group which has to be compatible with the supertranslations just as it is with the ordinary translations. For this we have the following general scheme, starting again at the infinitesimal level of the minimal approach:

Let $L^{(0)}$ be a complex finite-dimensional Lie algebra and $\pi^{(0,0)}$ a representation of $L^{(0)}$ in a complex vector space $M^{(0)}$. There are conjugations in $L^{(0)}$ and $M^{(0)}$ and $\pi^{(0,0)}$ is self-conjugate. Define the Lie algebra $P^{(0)}$ connected with the inhomogeneous transformations in $M^{(0)}$ as the semi-direct sum $P^{(0)}=M^{(0)}\oplus L^{(0)}$ with Lie bracket

$$[(u_1,h_1),(u_2,h_2)] = (\pi^{(0,0)}(h_1)u_2-\pi^{(0,0)}(h_2)u_1,[h_1,h_2]) \qquad \forall u_1,u_2\in M^{(0)},h_1,h_2\in L^{(0)} \tag{32}$$

We enlarge $P^{(0)}$ to a super Lie algebra P, or 'grade' $P^{(0)}$ as it is called in the physics literature, by extending the Abelian Lie algebra $M^{(0)}$ of ordinary translations to a supertranslation algebra $M=M^{(0)}\oplus M^{(1)}$. For this we choose a representation $\pi^{(1,1)}$ in a vector space $M^{(1)}$. This gives a direct sum representation $\pi=\pi^{(0,0)}\oplus\pi^{(1,1)}$ in $M^{(0)}\oplus M^{(1)}$. We then find, if possible, a symmetric bilinear map $s: M^{(1)}\times M^{(1)}\to M^{(0)}$ which is equivariant, i.e.

$$\pi^{(0,0)}(h)s(u_1,u_2) = s(\pi^{(1,1)}(h)u_1,u_2)+s(u_1,\pi^{(1,1)}(h)u_2) \qquad \forall h\in L^{(0)};\ u_1,u_2\in M^{(1)} \tag{33}$$

This turns M into a supertranslation algebra and $P=M\oplus L^{(0)}$ into a semi-direct sum super Lie algebra with graded bracket

$$[(u_1,h_1),(u_2,h_2)] = (-2is(u_1^{(1)},u_2^{(1)})-\pi(h_2)u_1+\pi(h_1)u_2,[h_1,h_2]) \qquad \forall u_1,u_2\in M;\ h_1,h_2\in L^{(0)} \tag{34}$$

A conjugation in $M^{(1)}$, self-conjugacy of $\pi^{(1,1)}$ and the reality condition for s are of course necessary ingredients for the definition of a proper conjugation in P.

We consider for a moment the even part of the situation only and note that the Lie group $\mathfrak{L}$ associated with $L_{sc}^{(0)}$ has an obvious infinite-dimensional representation in terms of functions on $M_{sc}^{(0)}$

$$(e^{th}\phi)(u) = \phi(e^{-t\pi^{(0,0)}(h)}u) \qquad \forall h\in L_{sc}^{(0)},u\in M_{sc}^{(0)} \tag{35}$$

If we let the functions be polynomials, the isomorphism between the algebra of polynomials on $M^{(0)}$ and the symmetric algebra over $M^{(0)\prime}$ allows us to reformulate (35) in a more algebraic manner. We then find that in this picture the Lie algebra $L^{(0)}$ is represented by 'one-particle' operators $-\sigma(\pi^{(0,0)}(h)')$. This suggests immediately a representation of $L^{(0)}$ in $S_{sc}(M')$ which can be combined with (26) to give a representation of the full super Lie algebra P. We call our earlier $E_L(u)$ or $E(u)$ of (26) $E(u,0)$ and define for every $(u,h)\in P$ the operators

$$E(u,h) = -A(u)+\frac{1}{2}\sigma(\tilde{ad}\,u)-\sigma(\pi(h)') \tag{36}$$

This defines indeed a representation of P in $S_{sc}^{\infty}(M')$ by derivations of degree $|(u,h)|$ and with the self-conjugacy property. Using the properties of σ given in part B of the appendix one checks in particular the commutation relations

$$[E(o,h_1),E(o,h_2) = [\sigma(\pi(h_1)'),\sigma(\pi(h_2)')]=\sigma([\pi(h_1)',\pi(h_2)'])= -\sigma([\pi(h_1),\pi(h_2)]')=E(0,[h_1,h_2]) \tag{37}$$

$$[E(o,h), E(u,o)] = [\sigma(\pi(h)'), A(u)] - \frac{1}{2}[\sigma(\pi(h)'), \pi(\tilde{ad}\, u)] =$$

$$= -A(\sigma(\pi(h)u)) + \frac{1}{2}\sigma([\pi(h), ad\, u]') =$$

$$= -A(\pi(h)u) + \frac{1}{2}\sigma(\tilde{ad}(\pi(h)u)) + E(\pi(h)u,o) \tag{38}$$

$\forall h, h_1, h_2 \in L^{(0)},\ u \in M$

The $E(u,h)$ are (partly) formal differential operators. Using again the basis $e_1^{(0)},.., e_m^{(0)}, e_1^{(1)}, \ldots, e_n^{(1)}$ in M one finds for the $E(e_j^{(\alpha)},0)$ precisely the $E_j^{(\alpha)}$ from formula (6) in Section II and with in particular the help of (64) for the $E(0,h)$ the expressions

$$E(o,h) = -\sigma(\pi(h)') = -\sum_{\alpha,\beta=0,1} C^{(\alpha)k}(\pi(h)')^{(\alpha,\beta)}{}_k{}^l A_l^{(\beta)} =$$

$$= x^k(\pi^{(0,0)}(h))_k{}^l \frac{\partial}{\partial x^l} + \theta^k(\pi^{(1,1)}(h))_k{}^l \frac{\partial}{\partial \theta^k} \tag{39}$$

With a further choice of a basis $f_1^{(0)}, \ldots, f_s^{(0)}$ in $L^{(0)}$, preferably with $f_j^{(0)*} = f_j^{(0)}$, this amounts to a formula for the generators $E(o, f_j^{(0)})$, $j = 1, \ldots, s$. The slightly unusual position of the indices of the matrices of the operators $\pi^{(\alpha,\alpha)}(h)$ comes from our definition of the matrix of an operator T in M as

$$T^{(\alpha,\beta)}{}_j{}^k = \langle Te_j^{(\alpha)}; e^{(\beta)k} \rangle \tag{40}$$

which is logical given our choice of pairing $M \times M' \to \mathbb{C}$ as a consequence of the pairing $V' \times V \to \mathbb{C}$ in the appendix. Note that with $u^{(\alpha)} = \xi^{(\alpha)j} e_j^{(\alpha)}$ the matrices $(\pi^{(\alpha,\alpha)}(h))_j{}^k$ acts on coordinates as $(\pi^{(\alpha,\alpha)}(h)u)^{(\alpha)j} = \xi^{(\alpha)j}(\pi^{(\alpha,\alpha)}(h))_j{}^k$. Writing things in this manner has the advantage that formulas remain valid when in the extension procedure $\mathbb{C}$ is replaced by $\mathfrak{B}$.

The restriction of E to $L_{sc}^{(0)}$ or $P_{sc}^{(0)} = M_{sc}^{(0)} \oplus L_{sc}^{(0)}$ can be exponentiated in $S_{sc}^{\infty}(M')$. Looking at elements ϕ of $S_{sc}^{\infty}(M')$ as C^{∞} functions from $M_{sc}^{(0)}$ to $\Lambda(M^{(1)\prime})$ one finds

$$(e^{E(u_1,h)}\phi)(u) = \Gamma(e^{-\pi^{(1,1)}(h)'})\phi(e^{-\pi^{(0,0)}(h)}(u - u_1 - \frac{1}{2}[u_1, u])) \tag{41}$$

for all $u, u_1 \in M_{sc}^{(0)}$, $h \in L_{sc}^{(0)}$; with $\pi^{(1,1)}(h)'$ in $M^{(1)\prime}$ the adjoint of the operator $\pi^{(1,1)}(h)$ in $M^{(1)}$ and $\Gamma(T)$ the operator in $\Lambda(M^{(1)\prime})$ obtained by tensoring the operator T in $M^{(1)\prime}$.

The full representation E of P_{sc} can of course not be exponentiated in $S_{sc}^{\infty}(M')$ but we have again a system $(P, S_{sc}^{\infty}(M'), E)$ which can ge extended by means of the scheme of Section III, to a system $(P^{\mathfrak{B}}, S_{sc}^{\infty}(M')^{\mathfrak{B}}, E^{\mathfrak{B}})$. Elements ϕ of $S_{sc}^{\infty}(M')^{\mathfrak{B}}$ determine functions from $(M^{\mathfrak{B}})_{sc}^{(0)}$ to $\mathfrak{B}$, C^{∞} in the sense given in Section IV. In terms of these functions the supergroups action associated with $(P^{\mathfrak{B}})_{sc}^{(0)}$ is generated simply by

$$(W(e^{(u_1,h)})\phi)(u) = (E^{E^{\mathfrak{B}}(u_1,h)}\phi)(u) =$$

$$= \phi(e^{-\pi^{\mathfrak{B}}(h)}(u-u_1-\frac{1}{2}[u_1,u])) \tag{42}$$

$$\forall (u_1,h)\in (P^{\mathfrak{B}})^{(0)}_{sc},\ u\in (M^{\mathfrak{B}})^{(0)}_{sc}$$

This representation and its minimal version as given by (36) might for obvious reasons be called the scalar superfield representation of P, but unfortunately the term has been given a different meaning in the physics literature.

A more general representation is obtained by choosing an additional representation ρ of $L^{(0)}$ in a finite-dimensional graded vector space F and taking in first instance $S^{\infty}_{sc}(M')\otimes F$ as representation space with operators

$$\tilde{E}(u,h) = 1\otimes \rho(h)+E(u,h)\otimes 1 \tag{43}$$

$S^{\infty}_{sc}(M')\otimes F$ is an $S^{\infty}_{sc}(M')$-module and the $\tilde{E}(u,h)$ are derivations in the sense of such modules, so we are in the more general situation briefly discussed in Section III, with a system $(P,S^{\infty}_{sc}(M'),S^{\infty}_{sc}(M')\otimes F,E)$ which can be extended to a system $(P^{\mathfrak{B}},S^{\infty}_{sc}(M')\otimes F)^{\mathfrak{B}},E^{\mathfrak{B}})$. Elements ϕ of the extended representation space $S^{\infty}_{sc}(M')\otimes F)^{\mathfrak{B}}$ can be regarded as functions from $M^{\mathfrak{B}})^{(0)}_{sc}$ to be $\mathfrak{B}$-module $F^{\mathfrak{B}}$, with the supergroup action now given by

$$W(e^{(u_1,h)})\phi(u) = \rho^{\mathfrak{B}}(h)\phi(e^{-\pi^{\mathfrak{B}}(h)}(u-u_1-\frac{1}{2}[u_1,u])) \quad \forall (u_1,h)\in (P^{\mathfrak{B}})^{(0)}_{sc};\ u\in (M^{\mathfrak{B}})^{(0)}_{sc} \tag{44}$$

At this point a general remark on our approach may be relevant. It will be clear that this section rests essentially on formula (26), the infinitesimal version of the (left) regular representation of the supertranslation group.

Other formulas like (36) and (43) are obtained from (26) by using obvious suggestions from the purely even case where one has ordinary group actions. We have introduced (26) as an Ansatz at the level of the minimal picture, which leads as we have shown in the extended picture to the results of Section II and more generally to an intuitively satisfactory notion of superfield in this and the next section. It is also possible to *derive* (26) and consequently (36) and (43) entirely within the minimal formalism, by using the theory of produced representations of Lie algebras, suitably adapted to the super case, as an infinitesimal version of induced representations of Lie groups. This can be found in a paper by Brussee[4]. We finally note that there is a covariance relation between the representation E of P and the right regular representation E_R of M. Combination of formulas (29), (30) and a modification of (38) leads to

$$[E(u_1,h),E_R(u_2)] = E_R(\pi(h)u_2) \quad \forall (u_1,h)\in P;\ u_2\in M \tag{45}$$

Let N be a subspace of M which is invariant under the representation π. Then $\{\phi\in S^{\infty}_{sc}(M')|E_R(u)\phi=0,\ \forall u\in N\}$ is invariant under E. In this manner the right regular representation of M is used in supersymmetric field theory to reduce the space of superfields, as we shall see in the next section.

VI. SUPERFIELDS IN 4-DIMENSIONAL SPACE-TIME

It is not hard to apply the results of the preceding section to the case of standard $N=1$ supersymmetric field theory in 4-dimensional space time: Let $L^{(0)}$ be the complexification of $so(1,3)$, which is of course isomorphic to the complexification of $sl(2,\mathbb{C})$ as a real Lie algebra. $M^{(0)}$ is the 4-dimensional space which carries as $\pi^{(0,0)}$ the vector representation $(1/2,1/2)$ of $so(1,3)$, while $M^{(1)}$ is also 4-dimensional and has as $\pi^{(1,1)}$ the spinor representation $(1/2,0)\oplus(0,1/2)$. $M^{(0)}$ and $M^{(1)}$ have invariant real structures. $M_{sc}^{(0)}$ is eventually identified with real Minkowski space-time and $M_{sc}^{(1)}$ is the real space of Majorana spinors. There is a unique bilinear map s: $M^{(1)}\times M^{(1)}\rightarrow M^{(0)}$, symmetric and with the required reality property. It corresponds to the well-known bilinear expression $\overline{\psi}_1^c\gamma^\mu\psi_2$, with $\overline{\psi}$ the Pauli adjoint of ψ and $\psi\mapsto\psi^c$ charge conjugation, here the invariant conjugation specifying $M_{sc}^{(1)}\subset M^{(1)}$. Putting all this together one obtains $P^{(0)}=M^{(0)}\oplus L^{(0)}$ as the (complexified) Poincaré algebra and finally $P=M\oplus L^{(0)}$ as the *super Poincaré algebra.*

Consider the representation E of P as given by formula (36). The elements ϕ of the representation space $S_{sc}^{\infty}(M')$ form what is usually called the *general superfield multiplet.* According to the discussion of Section IV the ϕ may be regarded as C^{∞} functions from $M_{sc}^{(0)}$, which is here real Minkowski space-time, to $\Lambda(M'^{(1)})$, the exterior algebra over the dual of the space of Dirac spinors. The space of general superfields in 4-dimensions is highly reducible. It has an interesting structure of invariant subspaces and it is at this point that understanding of the simple 1-dimensional model of Ref. 1 is no longer of much help. The occurrence and properties of these subspaces are intimately related to the possibility of formulating the typical $N=1$ supersymmetric models such as the standard self-interacting 'scalar' Wess-Zumino model and the 'scalar-vector' Wess-Zumino supergauge model, and are in particular related to the dynamical aspects of these models. We postpone a systematic discussion of the structure of $S_{sc}^{\infty}(M')$ and the more general representation spaces $S_{sc}^{\infty}(M')\otimes F$ of (43) together with their associated differential operators to a future paper where dynamical aspects of supersymmetric field theory will be treated and restrict ourselves here to a few remarks. For convenience we denote $S_{sc}^{\infty}(M')$ here as $\mathfrak{F}$. The basic observation is that the spinor representation in $M^{(1)}$ is reducible as a complex representation. We write $M=M^{+}\oplus M^{-}$, with M^{+} the space of the $(1/2,0)$ representation (the Weyl spinors) and M^{-} the space of the $(0,1/2)$ representation (the conjugate Weyl spinors). One has $(M^{+})^{*}=M^{-}$. Using our remark at the end of Section V, with formula (45), we define invariant subspaces $\mathfrak{F}^{\pm}$ of $\mathfrak{F}$

$$\mathfrak{F}^{\pm} = \{\phi\in\mathfrak{F}|E_R(u^{\mp})\phi=0,\ \forall u^{\mp}\in M^{\mp}\} \tag{46}$$

The $E_R(u^{\mp})$ are (odd) derivations and the $\mathfrak{F}^{\pm}$ are therefore subalgebras of $\mathfrak{F}$. These subalgebras are not invariant under conjugation, instead of this one has $(\mathfrak{F}^{\pm})^{*}=\mathfrak{F}^{\mp}$. The linear hull $\mathfrak{F}^{+}+\mathfrak{F}^{-}$ is an invariant subspace, self-conjugate and strictly smaller than $\mathfrak{F}$. It is almost a direct sum of $\mathfrak{F}^{+}$ and $\mathfrak{F}^{-}$ because the intersection $\mathfrak{F}^{+}\cap\mathfrak{F}^{-}$ is one-dimensional and equal to the subspace spanned by the unit element $e_{\mathcal{F}}$ of the algebra $\mathcal{F}$. The $\mathcal{F}^{\pm}$ can be shown to be isomorphic to the subalgebras $S_{sc}^{\infty}(M'^{(0)} \oplus M'^{\pm}) \cong C^{\infty}(M^{(0)};\ \Lambda(M'^{\pm}))$: There are automorphisms $\alpha^{\pm}$ of $\mathcal{F}$ with $\alpha^{+} = (\alpha^{-})^{-1}$, which have the property

$$(\alpha^{\pm})^{-1} E_R(u^{\mp}) \alpha^{\pm} = A(u^{\mp}) \qquad \forall u^{\mp} \in M^{\mp} \tag{47}$$

and therefore map $S_{sc}^{\infty}(M')^{(0)} \oplus M'^{\pm}$ onto $\mathcal{F}^{\pm}$. To check this and other properties and above all to clarify the connection with the more heuristic physical literature it is useful to choose again a basis in M consisting of vectors $e_{\mu}^{(0)}$, $\mu = 0,1,2,3$, in $M^{(0)}$, as in Sections IV and V, and of 'Weyl' vectors e_1^{+}, e_2^{+} in M^{+}, e_1^{-}, e_2^{-} in M^{-}, with $(e_j^{\pm})^{*} = -e_j^{\mp}$, $j = 1,2$, instead of the 'Majorana' vectors $e_k^{(1)}$ with $(e_k^{(1)})^{*} = -e_k^{(1)}$, $k = 1,2,3,4$. Correspondingly one has $e_{\mu}^{(0)}$ in $M'^{(0)}$ and $e^{\pm k}$, $j = 1,2$, in $M'^{\pm}$, with $(e^{\pm j})^{*} = e^{\pm j}$. A general superfield ϕ as an element of $S_{sc}^{\infty}(M') \cong C^{\infty}(M^{(0)};\ \Lambda(M'^{(1)}))$ may then be written as an expression similar to (21). The $E(e_{\mu}^{(0)},0)$, $E(e_j^{\mp},0)$ and $E_R(e_j^{\mp})$ are again (partly) formal differential operators, analogous to (5) and (6), with formal complex anticommuting variables θ^{+j}, θ^{-j}. The $E(e_j^{\mp},0)$ are the usual supersymmetry generators $Q, \bar{Q}$, the $E_R(e_j^{\mp})$ the *covariant derivatives* $D, \bar{D}$ and the $\theta^{\pm j}$ the usual variables $\theta, \bar{\theta}$. (All this up to convention dependent factors ± 1, $\pm i$). $S_{sc}^{\infty}(M'^{(0)} \oplus M'^{\pm})$ corresponds to the subspace of formal expressions in $S_{sc}^{\infty}(M')$ which are independent of the $\theta^{\mp j}$. The automorphisms $\alpha^{\pm}$ connecting $S_{sc}^{\infty}(m'^{(0)} \oplus M'^{\pm})$ with $\mathcal{F}^{\pm}$ are the unique automorphisms obtained by extension of the linear maps $M' \subset S(M') \to S(M')$

$$e^{(0)\mu} \mapsto e^{(0)\mu} \pm i s_{jk}^{+-\mu} e^{+j} e^{-k} \qquad e^{\pm j} \mapsto e^{\pm j} \tag{48}$$

with $s_{jk}^{+-\mu} e_{\mu}^{(0)} = s(e_j^{+}, e_k^{-})$. This basis is also helpful in reducing the representation of the super Poincaré algebra P with respect to subrepresentations of the ordinary Poincaré algebra $P^{(0)}$,i.e. exhibiting the various types of Poincaré component fields, scalars, vectors, spinors, etc. of which the multiplet consists.

The space $\mathcal{F}^{\pm}$ are the *chiral* (and anti-chiral) *multiplets*, also called *scalar multiplets.* They are used to formulate the standard self-interacting Wess-Zumino model. The general superfield space $\mathcal{F}$, or rather its self-conjugate part $\mathcal{F}_{sc}$, is sometimes called the *vector multiplet* because it contains a Poincaré vector field among its components. It is used in a theory in which it is couplet as a sort of supergauge potential to a pair of scalar multiplets. The subspace $(\mathcal{F}^{+} + \mathcal{F}^{-})_{sc}$ plays the rôle of a space of null-fields, with the physical or gauge invariant fields corresponding to elements of the quotient space $\mathcal{F}_{sc}/(\mathcal{F}^{+} + \mathcal{F}^{-})_{sc}$.

Finally it must be noted that in the physical literature one thinks of superfields in terms of what we have called the extended picture. This means that one does not use as basic superfield spaces $\mathcal{F}_{sc}$, $\mathcal{F}^{\pm}$, $(\mathcal{F}^{+} + \mathcal{F}^{-})_{sc}$, but instead of this what is in fact the result of the tensoring procedure discussed in Section III: $(\mathcal{F}^{\mathcal{B}})_{sc}^{(0)}$, $((\mathcal{F}^{\pm})^{\mathcal{B}})^{(0)}$, $((\mathcal{F}^{+} + \mathcal{F}^{-})^{\mathcal{B}})_{sc}^{(0)}$.

VII. Concluding remarks

In this paper we have given the first part of a rigorous and general mathematical framework for supersymmetric field theory, both in the extended and in the minimal approach, i.e. with and without explicit use of Grassmann variables. It is sufficient, as we have shown, for a proper definition of superfields in terms of representations of super Lie groups (the extended picture) or super Lie algebras (the minimal picture). We have not yet discussed Lagrangians, field equations, etc. in this general set-up. We also are still at the classical level, in the sense indicated in Section I of Ref. 1. Dynamical matters and quantization will be the subject of a future paper. Here we shall only sketch what use will be made of the results of this paper. For this we need a few general remarks on quantum field theory.

A *quantum field* is a linear map Ψ which assigns to every element ϕ of a linear space V of test functions an operator $\Psi(\phi)$ in a Hilbert space $\mathcal{H}$. For such a 'smeared' field operator one would think of the formula

$$\Psi(\phi) \text{ “=” } \int \Psi_j(x)\phi^j(x)d^4x \tag{49}$$

in which the right-hand side with the 'unsmeared' field operator $\Psi_j(x)$ has only heuristic meaning, even if subtleties connected with unbounded operators are ignored. There should be *covariance* with respect to a symmetry group $\mathcal{G}$ which contains the Poincaré group. This means that $\mathcal{G}$ acts in V by linear maps and that there is at the same time a unitary representation $U(\cdot)$ of $\mathcal{G}$ in V, which implements the action of $\mathcal{G}$ in V according to

$$\Psi(g\phi) = U(g)\Psi(\phi)U(g)^{-1} \qquad \forall g\in\mathcal{G};\ \phi\in V \tag{50}$$

Note that having $U(\cdot)$ in $\mathcal{H}$ implies having the dynamics, i.e. time evolution of the quantum field. There is an *infinitesimal* version of covariance. If one writes $U(e^b)=e^{\rho(b)}$, with b an element from the Lie algebra G of $\mathcal{G}$ and ρ a representation of G in $\mathcal{H}$ one has instead of (50)

$$\Psi(b\phi) = [\rho(b), \Psi(\phi)] \qquad \forall b\in G,\ \ \phi\in V \tag{51}$$

It is clearly this version that applies immediately to supersymmetry: Take $\mathcal{F}$, the space of classical superfields or a suitable subspace of $\mathcal{F}$, as test function space V. The super Poincaré algebra P acts by linear transformations in this V and plays the rôle of G. Constructing a supersymmetric quantum field theory, including its dynamics, means then finding a Hilbert space $\mathcal{H}$, field operators $\Psi(\phi)$, $\phi\in\mathcal{F}$, in $\mathcal{H}$ and a representation ρ of P in $\mathcal{H}$ such that the graded version of (51) holds. All this is in terms of the minimal approach. For a non-infinitesimal version of supersymmetry, corresponding to a formula like (50), one has to go over to the extended picture, with on one hand an extended test function space $V^{\mathcal{B}}=\mathcal{F}^{\mathcal{B}}$ and on the other hand an extended 'Hilbert space' $\mathcal{H}^{\mathcal{B}}=B\otimes_{\mathbf{C}}\mathcal{H}$, with field operators $\Psi^{\mathcal{B}}(\phi)$, $\phi\in V^{\mathcal{B}}$ and a representation $\rho^{\mathcal{B}}$ of $(P^{\mathcal{B}})^{(0)}_{sc}$ which can be exponentiated to a 'unitary' representation of the super Poincaré group in $\mathcal{H}^{\mathcal{B}}$, $\mathcal{H}$ is the space of physical states of the quantum system.

The extended space $\mathcal{H}^{\mathcal{B}}$ has of course no such interpretation. The main reason for considering nevertheless the extended picture in this context is that quantization procedures, such as the boson-fermion path integral, that give specific supersymmetric quantum theories are formulated in this picture. It is not easy to see what they mean in terms of the minimal picture, even in a heuristic sense. This seems to us an important and interesting problem. One of our tasks will be to clarify this situation for the simple model of Ref. 1. It is fully representative for the problem, but avoids the ordinary mathematical difficulties of quantum field theory, because it is still quantum mechanics.

VIII. Appendix

A. Graded vector spaces, algebras and Lie algebras

A vector space V over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ is called a $\mathbb{Z}_2$-graded or, as long as there is no other type of grading around, a *graded vector space* if V is given as a direct sum $V=V^{(0)}\oplus V^{(1)}$, in which $V^{(0)}$ is called the *even* part and $V^{(1)}$ the *odd* part of V. Elements v of $V^{(0)}$ and $V^{(1)}$ are called homogeneous. When different from 0 they have a *degree* denoted as $|v|$, with $|v|=0$ for $v\in V^{(0)}$ and $|v|=1$ for $v\in V^{(1)}$. Addition and multiplication of degree is modulo 2. A subspace W of V is called *graded subspace* whenever $W=(W\cap V^{(0)})\oplus(W\cap V^{(1)})$. In this case W has a natural grading compatible with that of V. A *linear map* T from a graded vector space V to a second graded vector space W has a degree $|T|$ when $TV^{(\alpha)}\subset W^{(\alpha+|T|)}$, $\alpha=0,1$. With this definition $\mathfrak{L}(V;W)$, the space of linear maps from V to W is also a graded vector space. Note that in the graded context we shall in general reserve terms as isomorphism, automorphism, homomorphism, etc. for *even* maps. The *dual* V' of V is graded in a natural manner, as is the tensor product $V_1\otimes\cdots\otimes V_k$ of graded vector spaces, the first according to $(V')^{(\alpha)}=\{u\in V'|<u;y>=0,\ \forall v\in V^{(\alpha+1)}\}$, the second with $|v_1\otimes\cdots\otimes v_k|=|v_1|+\cdots+|v_k|$. The last formula is only meaningful for elements v_j that are homogeneous and different from 0. This is obvious and will in general not be mentioned in similar situations.

The grading of the space of linear maps, duals and tensor products can be introduced in a more uniform way by specialization of the grading of the space $\mathfrak{L}(V_1,\cdots\cdots,V_k;W)$ of k-linear maps $V_1\times\cdots\times V_k\to W$. One has $V'=\mathfrak{L}(v;\mathbb{F})$, $\mathfrak{L}(V)=\mathfrak{L}(V;V)$ and in the finite dimensional case $V_1\otimes\cdots\otimes V_k\cong\mathfrak{L}(V_1',\ldots,V_k';\mathbb{F})$.

Algebras will be over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, associative and with unit element, unless there are statements to the contrary. A *graded* algebra A is an algebra which is graded as a vector space, $A=A^{(0)}\oplus A^{(1)}$, and with a multiplication that is compatible with the grading, which means $|ab|=|a|+|b|$. An obvious example of a graded algebra is $\mathfrak{L}(V)$, the space of linear operators in a graded vector space V. A second example is the tensor algebra $\Sigma_{k=0}^{\infty}\oplus(\otimes^k V)$ over V. A *derivation* D of degree $|D|$ of the algebra A is an element of $\mathfrak{L}(A)$ with $D(ab)=(Da)b+(-1)^{|D||a|}a(Db)$, $\forall a,\ b\in A$. In the purely even case derivations are 'infinitesimal automorphisms'. For the general graded case this is no longer true. An odd derivation cannot be exponentiated to an automorphism. A graded algebra A is called *graded commutative* if $ab=(-1)^{|a||b|}ba$. $\forall a, b\in A$. An example of a graded commutative algebra which is of central importance for the mathematical formulation of supersymmetry is the algebra of *graded symmetric tensors* over a graded vector space V. We denote it as $S(V)$ an discuss it separately below. Of even greater importance for the subject is the next concept:

A graded Lie algebra or *Lie superalgebra* L is a graded vector space $L=L^{(0)}\oplus L^{(1)}$, provided with a bilinear map $[\cdot,\cdot]$: $L\times L\to L$, the graded Lie bracket, with the following properties

$$[u,v]\in L^{(|u|+|v|)} \tag{52}$$

$$[u,v] = (-1)^{|u||v|+1}[v,u] \tag{53}$$

$$(-1)^{|u||w|}[u,[v,w]]+(-1)^{|v||u|}[v,[w,u]]+(-1)^{|w||v|}[w,[u,v]] = 0$$
$$\forall\ v,\ w\in L \tag{54}$$

An obvious example of a Lie superalgebra is again $\mathfrak{L}(V)$, the space of linear operators in a graded vector space V, with as graded Lie bracket the graded commulator $[T,S]=TS-(-1)^{|T||S|}ST$. A *representation* π of L in a graded vector space V is a homomorphism (of Lie superalgebras) of L into $\mathfrak{L}(V)$. A second important example of a Lie superalgebra is Der A, the space of derivations of a graded algebra, again with the graded commutator as graded Lie bracket.

B. The graded symmetric tensor algebra $S(V)$

Let V be a graded vector space, $A(V)=\Sigma_{k=0}^{\infty}\oplus(\otimes^k V)$, the algebra of tensors over V, and I_s the two-sided ideal generated by elements of the form $u\otimes v-(-1)^{|u||v|}v\otimes u$, with u and v homogeneous elements from V. We define $S(V)$, *the graded symmetric tensor algebra over* V, as the quotient algebra $S(V)=A(V)/I_s$. One checks easily that $S(V)$ is a graded commutative algebra in an obvious way. For the purely even case ($V=V^{(0)}$) this is the ordinary symmetric algebra and for the purely odd case ($V=V^{(1)}$) the exterior algebra. In the general case one has the linear isomorphism $S(V)\cong S(V^{(0)})\otimes\Lambda(V^{(1)})$, where the right-hand side is regarded as an object without grading. Elements of $S(V)$ can be written as finite sums of the unit element e and products $v_1^{(0)}\cdots v_p^{(0)}v_1^{(1)}\cdots v_q^{(1)}$ of homogeneous elements of V.

We use the following operators in $S(V)$:

1. Let $V'=(V')^{(0)}\oplus(V')^{(1)}=V^{(0)\prime}\oplus V^{(1)\prime}$ be the dual of V. The natural pairing between v' and V is written as $\langle\cdot,\cdot\rangle$: $V'\times V\to\mathbb{F}$. For every $u\in V'$ there is an $A(u)\in\mathfrak{L}(S(V))$ defined by

$$A(u)e = 0$$
$$A(u)v_1\cdots v_k = \sum_{j=1}^{k}(-1)^{(|v_1|+\cdots+|v_{j-1}|)|u|}\langle u;v_j\rangle v_1\cdots v_{j-1}v_{j+1}\cdots v_k \tag{55}$$

In stead of this one may use the more convenient formulas

$$A(u^{(0)})v_1^{(0)} \cdots v_p^{(0)}v_1^{(1)} \cdots v_q^{(1)} =$$
$$= \sum_{j=1}^{p} \langle u^{(0)};\, v_j^{(0)}\rangle v_1^{(0)} \cdots v_{j-1}^{(0)}v_{j+1}^{(0)} \cdots v_p^{(0)}v_1^{(1)} \cdots v_q^{(1)}$$
$$A(u^{(1)})v_1^{(0)} \cdots v_p^{(0)}v_1^{(1)} \cdots v_q^{(1)} =$$
$$= \sum_{j=1}^{p} (-1)^{j+1}\langle u^{(1)};\, v_j^{(1)}\rangle v_1^{(0)} \cdots v_p^{(0)}v_1^{(1)} \cdots v_{j-1}^{(1)}v_{j+1}^{(1)} \cdots v_q^{(1)} \tag{56}$$

2. For every $v \in V$ there is a $C(v) \in \mathfrak{L}(S(V))$ defined as

$$C(v)v_1 \cdots v_k = vv_1 \cdots v_k \tag{57}$$

The operator $A(u)$ is the unique *derivation* of degree $|u|$ of $S(V)$ with $A(u)v = \langle u;v\rangle e$. $C(v)$ is left multiplication in $S(V)$. $A(u)$ and $C(v)$ depend linearly on V' respectively V. They satisfy the *graded commutation relations*

$$\begin{aligned} &[A(u_1), |A(u_2)] = 0 \qquad [C(v_1),\, C(v_2)] = 0 \\ &[A(u),\, C(v)] = <u;v>1 \\ &\forall u, u_1, u_2 \in V';\ v, v_1, v_2 \in V. \end{aligned} \tag{58}$$

We may call $A(u)$ and $C(v)$ *annihilation* and *creation operators* because they appear as such in mixed Boson-Fermion Fock spaces in quantum many-particle theory. The mathematical situation is there however more special: An inner product in V connects V and V'. The $A(u)$ and $C(v)$ have also an interpretation as (formal) differentiation and multiplication operators. This is discussed in Section IV.

3. For every $T \in \mathfrak{L}(V)$, with degree $|T|$, there is a $\sigma(T) \in \mathfrak{L}(S(V))$ defined by

$$\sigma(T)e = 0$$
$$\sigma(T)v_1 \cdots v_k =$$
$$= \sum_{j=1}^{k} (-1)^{(|v_1| + \cdots + |v_{j-1}|)|T|} v_1 \cdots v_{j-1}(Tv_j)v_{j+1} \cdots v_k \tag{59}$$

The operator $\sigma(T)$ is the unique *derivation* of degree $|T|$ of $S(V)$ such that $\sigma(T)v = Tv$, $\forall v \in V$. (Note that instead of (59) we may use two formulas similar to (56).) It is easily verified that the following graded commutation relation is valid

$$[\sigma(T_1),\, \sigma(T_2)] = \sigma([T_1, T_2]) \qquad \forall T_1, T_2 \in \mathfrak{L}(V) \tag{60}$$

This means that σ is a representation of $\mathfrak{L}(V)$ as Lie superalgebra in $S(V)$. One has also

$$\begin{aligned} &[\sigma(T),\, C(v)] = C(Tv) \\ &[\sigma(T),\, A(u)] = -A(T'u) \end{aligned} \tag{61}$$

for all $T \in \mathfrak{L}(V)$, $v \in V$, $u \in V'$ and with $T' \in \mathfrak{L}(V')$ the *graded adjoint* of T defined by

$$\langle T'u;v\rangle = (-1)^{|T||u|}\langle u;Tv\rangle \qquad \forall u\in V',\ v\in V \tag{62}$$

The operator $\sigma(T)$ corresponds to what in the Fock space formalism is known as a *1-particle operator.* It is there sometimes denoted as $d\Gamma(T)$. In the finite-dimensional case it can be shown to be a bilinear expression in annihilation and creation operators. For this and for other purpose it may be useful tu choose a basis in V and obtain explicit *basis dependent* formulas:

Let V be (m,n) dimensional. Choose a basis in V, consisting as always of homogeneous elements, i.e. vectors $e_1^{(0)}, \ldots, e_m^{(0)}, e_1^{(1)}, \ldots, e_n^{(1)}$. In V' one then has the dual basis $e^{(0)1}, \ldots, e^{(0)m}, e^{(1)1}, \ldots, e^{(1)n}$ determined by $\langle e^{(\alpha)j}, e_k^{(\beta)}\rangle = \delta_{\alpha\beta}\delta_k^j$. This gives a basis in $S(V)$ consisting of the unit element e and the products $e_{i_1}^{(0)} \cdots e_{i_p}^{(0)} e_{j_1}^{(1)} \cdots e_{j_q}^{(1)}$, with $i_1 \leqslant \cdots \leqslant i_p$ and $j_1 < j_2 < \cdots < j_q$, $p=0,1,\ldots,\ q=0,1,\cdots n$.
An arbitrary element $\phi\in S(V)$ can be written as

$$\phi = \sum_{\substack{p=0,1,\cdots \\ q=0,1,\cdots n}} \frac{1}{p!}\,\frac{1}{q!}\, e_{i_1}^{(0)} \cdots e_{i_p}^{(0)} e_{j_1}^{(1)} \cdots e_{j_q}^{(1)}\, \alpha^{(p,q)i_1\cdots i_p;j_1\cdots j_q} \tag{63}$$

with $\alpha^{(p,q)i_1\cdots i_p j_1\cdots j_q}$ coefficients in $\mathbb{F}$, only a finite number different from 0, symmetric in the indices $i_1, \ldots, i_p$ and antisymmetric in the $j_1, \ldots, j_q$. We use the Einstein summation convention. The somewhat unusual way of writing the basis vectors in fronts of the coefficients is useful for further developments.

We have basis dependent annihilation operators $A^{(\alpha)k} = A(e^{(\alpha)k})$ and creation operators $C_k^{(\alpha)} = C(e_k^{(\alpha)})$, satisfying the relations $[A^{(\alpha)k}, A^{(\beta)l}]=0$ $[C_k^{(\alpha)}, C_l^{(\beta)}]=0$ and $[A^{(\alpha)k}, C_l^{(\beta)}] = \delta_{\alpha\beta}\delta_l^k \mathbf{1}$, $\alpha,\beta=0,1$. Using these one verifies easily that for an operator $T\in\mathfrak{L}(V)$, with matrix $T^{(\alpha,\beta)j}{}_k$, such that $Te_j^{(\beta)} = \Sigma_{\alpha=0,1} e_k^{(\alpha)} T^{(\alpha,\beta)k}{}_j$, one has for $\sigma(T)$ the explicit bilinear expression

$$\sigma(T) = \sum_{\alpha,\beta=0,1} C_k^{(\alpha)} T^{(\alpha,\beta)k}{}_j A^{(\beta)j} \tag{64}$$

C. Modules

Let A be an algebra over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. A *left module* over A is a vector space in which A acts from the left, i.e. there is a bilinear map $A\times V\to V$, denoted as $(a,\phi)\to a\phi$, with the properties

$$\begin{aligned} a(bv) &= (ab)v \\ (a+b)v &= av+bv \\ a(v_1+v_2) &= av_1+av_2 \\ ev &= v \qquad \forall a,\, b\in A;\ v, v_1, v_2\in V \end{aligned} \tag{65}$$

A *right module* over A has a right action $V\times A\to V$, $(v,a)\to va$, with similar properties. V is a *bimodule* over A if it is a left and right module and has the additional property

$$a(vb) = (av)b \qquad \forall a,b \in A\,; v \in V \tag{66}$$

REMARK. In the mathematical literature one usually considers modules over rings. The special case where A is an algebra gives some simplifications and is sufficient for our purpose.

Let A be a graded algebra. A (left, right) module V over A is a *graded module* if V is graded as a vector space and if the action of A in V is compatible with the grading in A and V, i.e. if $|av|$ (or $|va|)=|a|+|v|$, $\forall a \in A$, $v \in V$.

The most important case for us is that of a *graded module V over a graded commutative algebra A*. For commutative algebras it is hardly necessary to distinguish left and right modules. A left module is in a trivial way also a right module. This is not true for the general non-commutative case. For a graded commutative A we have in this respect basiccally the same situation as for a commutative A. If V is a left module over such an A it can be made immediately into a right module by defining $va=(-1)^{|a||v|}av$, $\forall a \in A$, $v \in V$. Left and right actions together satisfy (66), so V is also a bimodule. Consequently we are free to speak again in this case of a module, without further qualification, and we use its left or right character according to convenience, with a slight preference for writing actions from the left. For various derived concepts it pays however to keep a distinction between left and right objects, as may be clear from this paper.

MULTILINEAR ALGEBRA (multilinear maps, duals, tensor products, etc.) is a subject which carries over quite easily from vector spaces to modules over commutative algebras (or rings), as is clear from such standard expositions as Ref. 5 or on a more elementary level Ref. 6. In the general non-commutative situation multilinear algebra is a rather poor subject, but the graded commutative case is again almost a good as the commutative case. We shall in the following develop that part of linear algebra that we need in this and subsequent papers. In the remainder of this part of the appendix A will be a graded commulative algebra over $\mathbb{F}=\mathbb{R}$ of $\mathbb{C}$ and module will mean graded module over A. We start with general multilinear maps and derive from this various other concepts by specialization.

Let $V_1, \ldots, V_k$ and W be modules over A. A map $T\colon V_1 \times \cdots \times V_k \to W$, written as

$$(v_1, \ldots, v_k) \mapsto \langle T; v_1, \ldots, v_k\rangle \tag{67}$$

is called a (left) *k-linear map* (in the sense of A-modules) if it is k-linear over $\mathbb{F}$ and has the additional properties

$$\begin{aligned} &\langle T; v_1, \ldots, v_j a, v_{j+1}, \ldots, v_k \langle = \\ &= \langle T; v_1, \ldots, v_j, a v_j, \ldots, v_k\rangle \end{aligned} \tag{68}$$

$$\begin{aligned} &\langle T; v_1, \ldots, v_k a\rangle = \langle T; v_1, \ldots, v_k\rangle a \\ &\forall a \in A;\ j=1,2, \ldots, k-1;\ v_1, \ldots, v_k \in V \end{aligned} \tag{69}$$

Such a map has a degree $|T|=0$ or 1 if

$$\langle T;av_1,\ldots,v_k = (-a)^{|a||T|}a\langle T;v_1,\ldots,v_k\rangle \tag{70}$$

This together with the definition

$$\langle aT;v_1,\ldots,v_k\rangle = a\langle T;v_1,\ldots,v_k\rangle \tag{71}$$

makes $\mathfrak{L}_L(V_1,\ldots,V_k;W)$, the space of (left) k-linear maps, in a natural way into a module over A. (We shall sometimes write $\mathfrak{L}_L^A(V_1,\ldots,V_k;W)$ when we want to indicate that A is the basic ring of scalars.) Note that (69), (70) and (71) imply for the degrees

$$|\langle T;v_1,\ldots,v_k\rangle| = |T|+|v_1|+\cdots+|v_k| \tag{72}$$

Similarly one has a definition of right k-linear maps T written as

$$(v_1,\ldots,v_k)\mapsto\langle v_1,\ldots,v_k;T\rangle \tag{73}$$

with the properties

$$\langle v_1,\ldots,v_ja,v_{j+1},\ldots,v_k;T\rangle =$$

$$\langle v_1,\ldots,v_j,av_{j+1},\ldots,v_k;T\rangle \tag{74}$$

$$\langle av_l,\ldots,v_k;T\rangle = a\langle v_1,\ldots,v_k;T\rangle \tag{75}$$

The degree of such a T is defined by

$$\langle v_1,\ldots,v_ka,T\rangle = (-1)^{|a||T|}\langle v_1,\ldots,v_k;T\rangle a \tag{76}$$

and $\mathfrak{L}_R(V_1,\ldots,V_k;W)$, the space of right k-linear maps, becomes a module by defining

$$\langle v_1,\ldots,v_k;Ta\rangle = \langle v_1,\ldots,v_k;T\rangle a \tag{77}$$

Note that as collections of maps $V_1\times\cdots\times V_k\to W$ $\mathfrak{L}_L(V_l,\ldots,V_k;W)$ and $\mathfrak{L}_R(V_1,\ldots,V_k;W)$ are definitely distinct. They are nevertheless connected in a natural manner by an isomorphism of A-modules. Corresponding to $T\in\mathfrak{L}_L(V_1,\ldots,V_k;W)$ there is a $\hat{T}\in\mathfrak{L}_R(V_1,\ldots,V_k;W)$ defined by

$$\langle v_1,\ldots,v_k;\hat{T}\rangle = (-1)^{(|v_1|+\cdots+|v_k|)|T|}\langle T;v_1,\ldots,v_k\rangle \tag{78}$$

For $k=1$, $V_l=V$, we have two spaces of linear maps between the modules V and W, and specializing further to $W=V$ we obtain two distinct space of A-*linear operators* in V: $\mathfrak{L}_L(v)=\mathfrak{L}_L(V;V)$ and $\mathfrak{L}_R(V)=\mathfrak{L}_R(V;V)$. We have a preference for left acting operators, but right acting operators are also quite common in applications in supersymmetry (e.g. right acting Berezin differential operators $\dfrac{\overleftarrow{\partial}}{\partial\theta^j}$.)

For $k=1$, $V_1=V$ and $W=A$ we obtain two *duals* of V; the left dual $V_L{}'=\mathfrak{L}_L(V;A)$ and the right dual $V_R{}'=\mathfrak{L}_R(V;A)$, distinct as collection of maps from V to A, but isomorphic as A-modules.

The *tensor product* $V_l \otimes \cdots \otimes V_k$ is defined analoguously as for vector spaces, with due attention to degrees and corresponding additional minus signs. The result is an A-module with the two important properties

$$|v_1 \otimes \cdots \otimes v_k| = |v_1| + \cdots + |v_k| \tag{79}$$

$$v_1 \otimes \cdots \otimes a v_j \otimes \cdots \otimes v_k = (-1)^{|a|(|v_1| + \cdots + |v_{j-1})} a(v_1 \otimes \cdots \otimes v_k) \tag{80}$$

There is also an explicit construction of $V_1 \otimes \cdots \otimes V_k$ as (in general) a subset of $\mathfrak{L}_L((V_1)'_R, \ldots, (V_k)'_R A)$; or equivalently $\mathfrak{L}_R((V_l)'_L, \ldots, (V_k)'_L; A)$. In this a product $v_1 \otimes \cdots \otimes v_k$ is represented as

$$\langle v_1 \otimes \cdots \otimes v_k; u_1, \ldots, u_k \rangle =$$

$$=(-1)^{(|v_2| + \cdots + |v_k|)|k_1| + (|v_3| + \cdots + |v_k|)|u_2| + \cdots + |v_k||u_{k-1}|} \langle v_1; u_1 \rangle \cdots \langle v_k; u_k \rangle \tag{81}$$

with u_j elements of $(V_j)'_R$ and with the $v_j \in V_j$ identified with elements of the double dual $((V_j)'_R)'_L$.

A module is the generalization of a vector space in which one has as scalars the elements of an algebra (or ring) A instead of a field $\mathbb{F}$. One may in the same spirit generalize the concept of an algebra or a (super) Lie algebra over $\mathbb{F}$ to an algebra or (super) Lie algebra over a graded commutative algebra A. Such objects are in fact used in this paper and have obvious properties.

A module unlike a vector space may or may not have a *basis.* If it has a basis which in the graded case should consist of homogeneous elements, it is called a free module. If it has finite basis the number of even and odd elements is separately independent of the particular choice of this basis.

Let V be an (m,n) dimensional module over A, with basis $e_1^{(0)}, \ldots, e_m^{(0)}$, $e_1^{(1)}, \ldots, e_n^{(1)}$. An element $v \in V$ can be written in an unique manner as $v = a_L^{(0)i} e_i^{(0)} + a_L^{(1)i} e_i^{(1)}$ or as $v = e_i^{(0)} a_R^{(0)i} + e_i^{(1)} a_R^{(1)i}$. This means that the $e_i^{(\alpha)}$ can be used as right or left basis vectors. The $a_L^{(\alpha)i}$ and $a_R^{(\alpha)i} \in A^{(\alpha + |v|)}$ are *left*, respectively *right coordinates* of v with $a_L^{(0)i} = a_L^{(0)i}$, $a_L^{(1)i} = (-1)^{|v|} a_R^{(1)i}$. The left and right duals V_L' and V_R' are also (m,n) dimensional and one has the isomorphism $V \cong (V_L')_R' \cong (V_R')_L'$. The basis $\{e_i^{(\alpha)}\}$ gives dual bases $\{e_L^{(\alpha)i}\}$ in V_L' and $\{e_R^{(\alpha)i}\}$ in V_R' determined by $\langle e_L^{(\alpha)i}; e_j^{(\beta)} \rangle = \langle e_j^{(\beta)}; e_R^{(\alpha)i} \rangle = \delta_{\alpha\beta} \delta^i_j$. For left and right coordinates of v one has $a_L^{(\alpha)i} = \langle v; e_R^{(\alpha)i} \rangle$ and $a_R^{(\alpha)i} = \rangle e_L^{(\alpha)i}; v \rangle$. More generally one finds that $\mathfrak{L}_{L,R}(V_1, \ldots, V_k; W)$ has dimension $(\Sigma_{j=1}^k m_j + m$, $\Sigma_{j=1}^k n_j + n)$ if V_j and W have dimensions (m_j, n_j) and (m,n). There is an obvious basis determined by bases in V_j and W.

With respect to the basis $\{e_i^{(\alpha)}\}$ a (left) linear operator $T: V \rightarrow V$ has an A-valued *matrix* $T^{(\alpha,\beta)j}{}_k$ determined by $Te_k^{(\beta)} = \Sigma_{\alpha=0,1} e_j^{(\alpha)} T^{(\alpha,\beta)j}{}_k$. In terms of *right* coordinates it gives the usual transformation formula for $v' = Tv$: $a'^{(\alpha)j}_R = \Sigma_{\beta=0,1} T^{(\alpha,\beta)j}{}_k a_R^{(\beta)k}$. For a product of two operators one has the usual multiplication formula for the corresponding matrices.

By considering in V linear combinations of basis vectors $e_1^{(0)}, \ldots, e_m^{(0)}$, $e_1^{(1)}, \ldots, e_n^{(1)}$ with coefficients in $\mathbb{F}$ one obtains a subset $V_{\mathbb{F}}$ of V which is a vector space over $\mathbb{F}$. Linear operators in V with $\mathbb{F}$-valued matrices have restrictions to $\mathbb{F}$-linear operators in $V_{\mathbb{F}}$. V can in turn be recovered from $V_{\mathbb{F}}$ by multiplication with elements from A, i.e. as $A \otimes_{\mathbb{F}} V_{\mathbb{F}}$ or $V_{\mathbb{F}} \otimes_{\mathbb{F}} A$. For the first expression one uses the $\{e_j^{(\alpha)}\}$ as a right basis. $\mathbb{F}$-linear operators in $V_{\mathbb{F}}$ extend to special A-linear operators in V. Arbitrary operators in V are linear combinations of these with coefficients in A. The idea of isolating a $V_{\mathbb{F}}$ from V in this manner is important in Section II, the reverse process of obtaining V and its operators from $V_{\mathbb{F}}$ is essential for the subsequent sections.

D. Conjugations

In our discussion of supersymmetry in this paper we have situations that are *real*, in the sense of being represented by *real substructures* of *complex systems.* The reasons for this are given in Section VI. This means that the basic field $\mathbb{F}$ is $\mathbb{C}$ and that all objects have mutually consistent *conjugations.* A conjugation of a vector space V over C is a antilinear map $V \to V$, denoted as $v \mapsto v^*$, or $v \mapsto Cv$, or also sometimes as $v \mapsto v^C$, with $(v^*)^* = v$. The real subspace of self-conjugate elements is denoted as V_{sc}. If V is a graded vector space then a conjugation must be *even.* A conjugation in V defines a conjugation in the dual V' by

$$\langle u^*;v\rangle = (-1)^{|u||v|}\langle u;v^*\rangle \qquad \forall u \in V',\ v \in V \tag{82}$$

A conjugation in an (associative) algebra A is supposed to have the additional property $(ab)^* = b^* a^*$, $\forall a,\ b \in A$. In a super Lie algebra L we require

$$[v_1,v_2]^* = (-1)^{|v_1||v_2|}[v_1^*,v_2^*] \qquad \forall v_1,\ v_2 \in L \tag{83}$$

A representation π of a super Lie algebra with conjugation in a graded vector space V with conjugation is called self-conjugate if

$$(\pi(v)\phi)^* = (-1)^{|v||\phi|}\pi(v^*)\phi^* \qquad \forall v \in L,\ \phi \in V \tag{84}$$

For a conjugation in a (graded) module V over a (graded commutative) algebra A with conjugation one should have

$$(av)^* = (-1)^{|a||v|}a^* v^* \qquad \forall a \in A,\ v \in V \tag{85}$$

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REFERENCES

1. P.J.M. BONGAARTS. Mathematical aspects of supersymmetric theories. I. (This volume)
2. A. ROGERS (1981). *J. Math. Phys. 22,* 939.
3. A. ROGERS (1986). *Commun. Math. Phys. 105,* 375.

4. R. BRUSSEE (1989).Ann. Inst. Henri Poincaré 50, 1.
5. N. BOURBAKI (1948). *Eléments de Mathématique VII, Livre II. Algèbre,* Chapitre III. Paris.
6. T.S. BLYTH (1977). Module theory. Oxford.

Theory of a charged spinning particle in a gravitational and electromagnetic field*

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Abstract
The classical and quantum mechanics of a charged spinning particle, such as an electron, in arbitrary gravitational and electromagnetic background fields is presented. The classical theory possesses supersymmetry. The equivalent Dirac formulation of the quantum theory is derived.

1. Classical mechanics of the spinning particle

A relativistic pointlike fermion, a particle with spin $s = 1/2$, having mass m and electric charge q, can be described in the classical limit $h \rightarrow 0$ by a Lagrangian

$$L = \frac{m}{2} g_{\mu\nu}(x)\, \dot{x}^{\mu} \dot{x}^{\nu} + \frac{i}{2} \psi_a \frac{D\psi^a}{D\tau} + q(A_{\mu}(x)\, \dot{x}^{\mu} - \frac{i}{2m} F_{ab}(x)\, \psi^a \psi^b). \tag{1}$$

Indeed, in this lecture I want to show that quantization of the theory defined by L leads to the well-known Dirac theory of an electron in background electromagnetic and gravitational fields specified by A_{μ} and $g_{\mu\nu}$.

The notation in eq.(1) is as follows. The $x^{\mu}(\tau)$ are the particle's space-time co-ordinates, a dot denoting a derivative with respect to the worldline proper-time τ. The ψ^a are anti-commuting c-numbers (Grassmann variables) transforming as a vector under local Lorentz transformations, i.e. under $SO(3,1)$ rotations of the local pseudo-orthogonal (Minkowskian) co-ordinate system $\xi^a(\tau)$ at the particle's instantaneous position given by $x^{\mu}(\tau)$. These co-ordinates are related by the differential condition

$$d\xi^a = e^a_{\mu}(x) dx^{\mu}, \tag{2}$$

with the infinitesimal, line-element being defined by

$$ds^2 = \eta_{ab} d\xi^a d\xi^b = g_{\mu\nu}(x) dx^{\mu} dx^{\nu}. \tag{3}$$

Here η_{ab} is the constant Minkowski metric of flat space-time. From now on I will choose $\eta_{ab} = \delta_{ab}$, with the 4th (time-like) co-ordinate being imaginary, so there is no longer a need to distinguish between upper and lower indices on locally Lorentzian vector- and tensor components. The co-efficients $e^a_{\mu}(x)$ are the vierbein variables, with the inverse denoted by $e^{a\mu}(x)$. According to the metric postulate they satisfy

$$D_{\mu} e^a_{\nu} = \partial_{\mu} e^a_{\nu} - \omega^{ab}_{\mu} e^b_{\nu} - \Gamma^{\lambda}_{\mu\nu} e^a_{\lambda} = 0. \tag{4}$$

The $\Gamma^{\lambda}_{\mu\nu}$ are the components of the Riemann-Christoffel affine connection, obtained in terms of the vierbein by inverting eq.(4), and the ω^{ab}_{μ}, which are anti-symmetric in a and b, denote the components of the spin-connection. Their explicit form can be found in the absence of torsion by solving the anti-symmetric part of eq.(4):

$$D_{[\mu} e^a_{\nu]} = \partial_{[\mu} e^a_{\nu]} - \omega^{ab}_{[\mu} e^b_{\nu]} = 0. \tag{5}$$

The covariant derivative of ψ^a is defined by

$$\frac{D\psi^a}{D\tau} = \dot{\psi}^a - \dot{x}^{\mu} \omega^{ab}_{\mu} \psi^b. \tag{6}$$

The function $A_{\mu}(x)$ represents the electromagnetic vector potential and F^{ab} is the locally Lorentzian form of the field strength:

$$F^{ab}(x) = e^{a\mu} e^{b\nu} (\partial_{\mu} A_{\nu}(x) - \partial_{\nu} A_{\mu}(x)). \tag{7}$$

Finally, the Grassmann co-ordinates are taken to be real under complex conjugation, although their order is reversed:

$$(\psi^a)^* = \psi^a, \quad (\psi^{a_1} \cdots \psi^{a_k})^* = \psi^{a_k} \cdots \psi^{a_1}. \tag{8}$$

The reality of the Lagrangian L then requires a factor i in front of the terms which are quadratic in the anti-commuting variables in eq.(1).

Requiring the action (the τ-integral of L) to be stationary with respect to variations of the independent variables (x^μ, ψ^a), keeping the end-points fixed, one obtains the equations of motion:

$$m g_{\mu\nu} \frac{D^2 x^\nu}{D\tau^2} = q F_{\mu\nu} \dot{x}^\nu - \frac{i}{2} R^{ab}_{\mu\nu} \dot{x}^\nu \psi^a \psi^b - \frac{iq}{2m} D_\mu F^{ab} \psi^a \psi^b \tag{9}$$

$$\frac{D\psi^a}{D\tau} = \frac{q}{m} F^{ab} \psi^b. \tag{10}$$

The various as yet undefined quantities in these equations have the following meaning:

$$\frac{D^2 x^\mu}{D\tau^2} = \ddot{x}^\mu + \Gamma^\mu_{\lambda\nu} \dot{x}^\lambda \dot{x}^\nu, \tag{11}$$

is the second covariant derivative of x^μ, whilst $R^{ab}_{\mu\nu}$ is the curvature tensor expressed in terms of the spin-connection:

$$R^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu - [\omega_\mu, \omega_\nu]^{ab}; \tag{12}$$

furthermore, the covariant derivative of the electromagnetic field strength reads

$$D_\mu F^{ab} = \partial_\mu F^{ab} - [\omega_\mu, F]^{ab}. \tag{13}$$

The Lagrangian L is invariant modulo a total τ-derivative under several sets of infinitesimal transformations, to wit:
-proper-time translations:

$$\delta x^\mu = \xi \dot{x}^\mu, \quad \delta\psi^a = \xi \dot{\psi}^a, \tag{14}$$

ξ being the constant parameter of the infinitesimal transformations;
-supersymmetry transformations:

$$\begin{aligned} \delta x^\mu &= -i\epsilon\, \psi^a e^{a\mu} \equiv -i\epsilon \psi^\mu(x), \\ \delta\psi^\mu(x) &= m\dot{x}^\mu \epsilon, \end{aligned} \tag{15}$$

with ϵ a constant anti-commuting parameter; the last equation is actually equivalent with the following covariant transformation of ψ^a:

$$\Delta\psi^a \equiv \delta\psi^a - \delta x^\mu \omega^{ab}_\mu \psi^b = m e^a_\mu \dot{x}^\mu \epsilon; \tag{16}$$

-chiral transformations with parameter α:

$$\delta x^\mu = 0, \quad \delta\psi^a = \frac{i\alpha}{3!} \epsilon^{abcd} \psi^b \psi^c \psi^d. \tag{17}$$

These symmetries play an important role in clarifying the spectrum of states of the quantum theory, as we will see.

2. CANONICAL STRUCTURE OF THE THEORY

Now that we have a Lagrangian $L(x^\mu,\psi^a)$ with the symmetries given in eqs.(14-17), and equations of motion (9,10), we can derive the canonical phase-space structure of the theory. We begin with the canonical momenta:

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = m g_{\mu\nu}\dot{x}^\nu + qA_\mu - \frac{i}{2}\omega_\mu^{ab}\psi^a\psi^b, \tag{18}$$

$$\pi^a = \frac{\partial L}{\partial \dot{\psi}^a} = -\frac{i}{2}\psi^a. \tag{19}$$

From the equation for the anti-commuting momentum π^a it is clear, that the usual canonical methods cannot apply directly to the Grassmann variables: we cannot solve for the velocities $\dot{\psi}^a$ in terms of the momenta π^a. Indeed, there is an algebraic relation between the co-ordinates ψ^a and their momenta π^a:

$$\chi^a = \pi^a + \frac{i}{2}\psi^a = 0. \tag{20}$$

Thus, the momenta and co-ordinates form a linearly dependent set of variables and independent variation of all of them, as required for instance by Hamilton's principle or in the definition of Poisson brackets, is not consistent with the dynamics as expressed by the constraints $\chi^a = 0$. Obviously this results from the fact, that the Lagrangian is only linear in the time derivative of ψ^a, and not quadratic like for x^μ; equivalently, the equation of motion for ψ^a is only of first order in the τ-derivative and admits only one constant of integration, whereas the equation for x^μ is a second-order differential equation and admits two constants of integration. The result of this linear dependence is therefore, that the canonical phase space of the Grassmann co-ordinates is only half as large as one expects, being parametrized completely by the ψ^a.

The way to solve this difficulty is to introduce Dirac brackets instead of the usual Poisson brackets, defined by

$$\{A,B\}^* = \frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial p_\mu} - \frac{\partial A}{\partial p_\mu}\frac{\partial B}{\partial x^\mu} + i(-)^A\frac{\partial A}{\partial \psi^a}\frac{\partial B}{\partial \psi^a} \tag{21}$$

for any functions $A(x,p,\psi), B(x,p,\psi)$, where all dependence on the momenta π^a has been removed by application of the constraints $\chi^a = 0$. Here the symbol $(-)^A$ denotes the Grassmann parity of the quantity A: $+1$ if A is even and -1 if A is odd. The definition (21) implies in particular the following elementary Dirac brackets:

$$\{x^\mu,p_\nu\}^* = \delta_\nu^\mu, \quad \{\psi^a,\psi^b\}^* = -i\delta^{ab}. \tag{22}$$

With these definitions the equations of motion can be written in canonical form as

$$\frac{dA}{d\tau} = \frac{\partial A}{\partial \tau} - \{H,A\}^*, \tag{23}$$

where H is the Hamiltonian:

$$H = \dot{x}^{\mu} p_{\mu} + \dot{\psi}^{a} \pi^{a} - L \tag{24}$$
$$= \frac{1}{2m} g^{\mu\nu} \Pi_{\mu} \Pi_{\nu} + \frac{iq}{2m} F^{ab} \psi^{a} \psi^{b},$$

and where Π_{μ} is defined by

$$\Pi_{\mu} = p_{\mu} + \frac{i}{2} \omega_{\mu}^{ab} \psi^{a} \psi^{b} - q\, A_{\mu} \tag{25}$$
$$= m g_{\mu\nu} \dot{x}^{\nu}.$$

Eq.(23) implies, that the Hamiltonian H acts as the generator of the proper-time translations of eqs.(14). Similarly, we can construct conserved quantities ('Noether charges') generating the supersymmetry and chiral transformations of eqs.(15-17). Indeed, Noether's theorem asserts, that if the Lagrangian L is invariant up to a total derivative under transformations $(\delta x^{\mu}, \delta\psi^{a})$:

$$\delta L = \frac{dB}{d\tau}, \tag{26}$$

then the quantity

$$G = \delta x^{\mu} \frac{\partial L}{\partial \dot{x}^{\mu}} + \delta\psi^{a} \frac{\partial L}{\partial \dot{\psi}^{a}} - B \tag{27}$$

is such a conserved charge, the equations of motion implying

$$\frac{dG}{d\tau} = 0. \tag{28}$$

Using this algorithm we find, that the supersymmetry transformations (15) are generated by a conserved charge

$$Q = \Pi_{\mu} \psi^{\mu} \tag{29}$$
$$= e^{a\mu} \left(p_{\mu} + \frac{i}{2} \omega_{\mu}^{bc} \psi^{b} \psi^{c} - q\, A_{\mu} \right) \psi^{a}.$$

Hence the supersymmetry transformation of any arbitrary function $A(x,p,\psi)$ is given by

$$\delta_{\epsilon} A = i\epsilon \{Q, A\}^{*}. \tag{30}$$

In a similar way, chiral transformations are generated by the conserved charge

$$\Gamma_{*} = \frac{1}{4!} \epsilon^{abcd} \psi^{a} \psi^{b} \psi^{c} \psi^{d}, \tag{31}$$

with the transformations (17) generalizing for functions $A(x,p,\psi)$ to

$$\delta_{\alpha} A = \alpha \{\Gamma_{*}, A\}^{*}. \tag{32}$$

The generator Γ_{*} is Grassmann-even, therefore it is nilpotent under the Dirac-bracket operation[1]:

1. However, observe that the same is true in odd dimensions, in which Γ_{*} is Grassmann-odd.

$$\{\Gamma_*,\Gamma_*\}^* = 0. \tag{33}$$

In contrast, the supersymmetry charge is Grassmann-odd, and its Dirac-bracket gives the Hamiltonian:

$$\{Q,Q\}^* = -2imH. \tag{34}$$

This is the classical equivalent of the standard quantum mechanical supersymmetry algebra (to be obtained in the next section), which by definition contains a generator Q acting as the square root of the Hamiltonian. Of course, the conservation of the Noether charges, eq.(28), implies that Q and Γ_* commute with the Hamiltonian in the sense of Dirac-brackets:

$$\{H,Q\}^* = 0, \quad \{H,\Gamma_*\}^* = 0. \tag{35}$$

This completes the present analysis of the canonical structure of the classical theory of the spinning particle. Now we turn to the quantization of this theory.

3. Quantum theory of the spinning particle

The operator formulation of the quantum mechanical theory of the spinning particle is obtained according to the correspondence principle by replacing the dynamical variables (x,p,ψ) by corresponding operators (X,P,Γ), satisfying (anti-)commutation relations:

$$[X^\mu,P_\nu] = i\delta^\mu_\nu, \quad \{\Gamma^a,\Gamma^b\} = \delta^{ab}. \tag{36}$$

Here the square brackets denote a commutator and the curly braces an anti-commutator as usual. Expressions for composite operators, like the Hamiltonian, sometimes suffer from operator ordering ambiguities. These can be resolved by requiring the supercharge Q to be hermitean:

$$Q = \frac{1}{2}(\Pi_\mu\Gamma^\mu + \Gamma^\mu\Pi_\mu) = \Gamma^\mu\mathcal{P}_\mu, \tag{37}$$

with

$$\mathcal{P}_\mu = \Pi_\mu + \frac{i}{2}\Gamma^\nu_{\mu\nu} = \sqrt[4]{g}\,\Pi_\mu\frac{1}{\sqrt[4]{g}}. \tag{38}$$

In this expression Π_μ is the operator version of eq.(25), which is hermitean and has no operator-ordering problems because of the anti-symmetry of ω_μ^{ab}, whilst $\Gamma^\mu(X)$ is Γ^a contracted with a vierbein operator, analogous to $\psi^\mu(x)$ in eq.(15). Then an unambiguous expression for the Hamiltonian is obtained by requiring it to satify the quantum analogue of the supersymmetry algebra (34):

$$H = \frac{1}{2m}\{Q,Q\}. \tag{39}$$

This Hamiltonian is guaranteed to be hermitean. In order to write it in a convenient form, let

$$\Sigma^{\mu\nu} = \frac{1}{2}[\Gamma^\mu,\Gamma^\nu], \tag{40}$$

and

$$G_{\mu\nu} = [\mathcal{P}_\mu, \mathcal{P}_\nu]; \tag{41}$$

then the Hamiltonian can be written as

$$H = \frac{1}{2m}(\mathcal{P}^2 + \Sigma^{\mu\nu} G_{\mu\nu}), \tag{42}$$

where the first term on the right hand side is in components:

$$\mathcal{P}^2 \equiv \mathcal{P}_\mu \mathcal{P}^\mu + \Gamma^\mu_{\mu\nu} \mathcal{P}^\nu, \quad \mathcal{P}^\nu = g^{\nu\lambda} \mathcal{P}_\lambda. \tag{43}$$

Observe the importance of ordering in the last expressions.

It is instructive to construct an explicit representation of the operator algebra (36), a particularly useful one being the co-ordinate representation, defined by

$$\begin{aligned} X^\mu &\rightarrow x^\mu, \\ P_\mu &\rightarrow -i \frac{1}{\sqrt[4]{g}} \partial_\mu \sqrt[4]{g} = -i\partial_\mu - \frac{i}{2}\Gamma^\nu_{\mu\nu}, \\ \Gamma^a &\rightarrow \frac{1}{\sqrt{2}} \gamma^a, \end{aligned} \tag{44}$$

where in $(3+1)$ space-time dimensions the γ^a are the usual (hermitean) 4×4 Dirac matrices, satisfying

$$\{\gamma^a, \gamma^b\} = 2\,\delta^{ab}. \tag{45}$$

The representation of P_μ is precisely as required by hermiticity of the gradient operator with respect to the covariant inner product

$$(\phi(x), \psi(x)) = \int d^n x \sqrt{g}\, \phi^*(x)\, \psi(x). \tag{46}$$

In the representation (44), $\mathcal{P}_\mu$ is $-i$ times the covariant derivative:

$$\mathcal{P}_\mu = -iD_\mu = -i(\partial_\mu - iq\, A_\mu - \frac{1}{2}\omega_\mu^{ab} \sigma^{ab}), \tag{47}$$

with σ^{ab} the generators of the Lorentzgroup in the spinor repreentation:

$$\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]. \tag{48}$$

Then the supercharge becomes the Dirac operator

$$Q = -\frac{i}{\sqrt{2}} \gamma \cdot D, \tag{49}$$

and the Hamiltonian the corresponding covariant Laplacian:

$$H = -\frac{1}{2m}(\mathcal{D}^2 + \frac{1}{4}R - iq\sigma^{ab} F_{ab}). \tag{50}$$

Here the operator $\mathcal{D}^2$ is the direct analogue of $\mathcal{P}^2$ in eq.(43):

$$\mathcal{D}^2 = (D_\mu + \Gamma^\lambda_{\lambda\mu}) g^{\mu\nu} D_\nu. \tag{51}$$

Furthermore, $R = e^{a\mu} e^{b\nu} R^{ab}_{\mu\nu}$ is the Riemann curvature scalar. The term $\sigma^{\mu\nu}[D_\mu, D_\nu]$, corresponding to $\Sigma^{\mu\nu} G_{\mu\nu}$ in eq.(42), has been evaluated using the Ricci identity:

$$[D_\mu, D_\nu] = -iqF_{\mu\nu} - \frac{1}{2} R^{ab}_{\mu\nu} \sigma^{ab}, \tag{52}$$

and the Bianchi identities for $R_{\mu\nu\kappa\lambda} = R^{ab}_{\mu\nu} e^a_\kappa e^b_\lambda$:

$$R_{\mu[\nu\kappa\lambda]} = 0, \;\; R_{\mu\nu\kappa\lambda} = R_{\kappa\lambda\mu\nu}, \tag{53}$$

together with the Dirac-matrix identity

$$\{\sigma^{ab}, \sigma^{cd}\} = \frac{1}{2}(\epsilon^{abcd}\gamma_5 + \delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}). \tag{54}$$

Finally, in this representation the operator Γ_* is

$$\Gamma_* = \frac{1}{4}\gamma_5 \tag{55}$$

It commutes with the Hamiltonian, and anti-commutes with the supercharge

$$[H, \gamma_5] = 0, \quad \{Q, \gamma_5\} = 0, \tag{56}$$

whilst

$$(\gamma_5)^2 = 1, \tag{57}$$

in contrast with the classical equation (33).

4. The spectrum of states in the quantum theory

In the representation (44), the states of the spinning particle are described by wavefunctions transforming as spinors under local Lorentz transformations. These wavefunctions are solutions of the Schrödinger equation:

$$i\partial_\tau \Phi(x) = H\Phi(x) = \epsilon\, \Phi(x), \tag{58}$$

where the last equality holds for stationary states only. Note that because H is the square of a hermitean operator, eq.(39), its eigenvalues are nonnegative: $\epsilon \geqslant 0$. Since the Hamiltonian commutes with γ_5, we can diagonalize γ_5 and obtain chiral solutions

$$H\Phi_\pm = \epsilon\, \Phi_\pm, \;\; \gamma_5 \Phi_\pm = \pm\Phi_\pm. \tag{59}$$

On the other hand, Q anticommutes with γ_5, hence it changes the chirality of any eigenstate of γ_5. Hence Q cannot have diagonal elements in this basis, and

$$Q\Phi_+ = \lambda\chi_-, \tag{60}$$

with χ_- a state of chirality opposite to that of Φ_+, and λ is a normalization factor to be chosen to suit convenience. Now we must distinguish two different cases:

1. $\lambda \neq 0$. Then it follows from (60), that

$$i\gamma \cdot D\Phi_+ = -\lambda\sqrt{2}\chi_-, \quad i\gamma \cdot D\chi_- = -\frac{m\sqrt{2}}{\lambda}H\Phi_+. \tag{61}$$

Therefore, by choosing $\lambda^2 = m\epsilon$, and renaming $\chi_- = \Phi_-$, we obtain

$$i\gamma \cdot D\Phi_+ = -\sqrt{2m\epsilon}\,\Phi_-, \quad i\gamma \cdot D\Phi_- = -\sqrt{2m\epsilon}\,\Phi_+. \tag{62}$$

Thus we have pairs of states of opposite chirality $\Phi_\pm$, belonging together, with the same energy eigenvalue ϵ, and forming a Dirac spinor $\Phi = \Phi_+ + \Phi_-$ which satisfies the Dirac equation

$$(i\gamma \cdot D + M)\Phi = 0. \tag{63}$$

Here $M = \sqrt{2m\epsilon}$, which equals the classical particle mass m only if $\epsilon = m/2$.

2. $\lambda = 0$. In this case we have no connection between solutions of different chirality, because the states are annihilated by Q (i.e. they are invariant under supersymmetry):

$$i\gamma \cdot D\Phi_\pm = 0. \tag{64}$$

Since, as observed above, the Hamiltonian is non-negative, these zero-modes of the Dirac operator constitute the absolute minima of the energy spectrum $\{\epsilon\}$. Furthermore, because they correspond to singlet representations of supersymmetry, there may exist different numbers of solutions with positive and negative chirality. This is to be contrasted to the case of massive particles ($M \neq 0$), which always involves pairs of wavefunctions of opposite chirality. The difference in the number of linearly independent positive and negative chirality zero-modes of the supercharge Q:

$$\Delta = n_+^0 - n_-^0, \tag{65}$$

is known as the Witten index, and co-incides in this case with the index of the Dirac operator $i\gamma \cdot D$. This index is a topological invariant of the spacetime manifold on which the particle moves, thus being stable under small perturbations of the potentials A_μ and $g_{\mu\nu}$. Because the non-zero modes occur in pairs of opposite chirality, a convenient way to compute the index is by taking the trace of a regularized γ_5-operator over all of the Hilbert space:

$$\Delta = Tr\ \gamma_5 \exp -\beta H. \tag{66}$$

Only contributions from states with $\epsilon \neq 0$ can give rise to β-dependent terms in the trace, but those cancel among themselves, leaving only the β-independent zero modes to contribute to a non-vanishing value of the index Δ. Conversely, if the index has a non-zero value there exists at least one (normalizable) zero-mode of the supersymmetry operator Q, and therefore the groundstate(s) of the system is (are) supersymmetric:

$$Q\Psi_0 = i\gamma \cdot D\Psi_0 = 0. \tag{67}$$

Therefore this statement is independent of the detailed form of the background fields A_μ and $g_{\mu\nu}$. Finally it should be mentioned, that the index of the Dirac operator gives the value of the chiral anomaly, which plays an important role in the analysis of quantum gauge theories. For details of these applications the reader is referred to the literature.

REFERENCES

The following list, though far from exhaustive, gives the main references to the literature on spinning particles and supersymmetric quantum mechanics. Further references may be found in the papers quoted.

Spinning particle models were constructed in:

[1] A. BARDUCCI, R. CASALBUONI and L. LUSANNA (1976). *Nuovo Cim.* 35A, 377.

[2] L. BRINK, P.DI VECCHIA and P. HOWE (1977). *Nucl. Phys.* B118, 76.

[3] P. SALOMONSON (1978). *Phys. Rev.* D18, 1868.

Supersymmetric quantum mechanics has been studied in

[4] E. WITTEN (1981). *Nucl. Phys.* B188, 513.

[5] P. SALOMONSON and J.W. VAN HOLTEN, *Nucl. Phys.* B196 (1982), 509.

[6] M. DE CROMBRUGGE and V. RITTENBERG (1983) *Ann. Phys. 151, 99.*

The canonical analysis of supersymmetric non-linear σ-models (with $N=2$) and spinning particles in a gravitational field has been performed in

[7] A.C. DAVIS, A.J. MACFARLANE, P. POPAT and J.W. VAN HOLTEN (1984). *J. Phys.* A17, 2945.

The relation of supersymmetric quantum mechanics with index theorems has been established in

[8] E. WITTEN (1982). *Nucl. Phys.* B202, 253.

and studied further in

[9] L. ALVAREZ-GAUMÉ (1983). *Comm. Math. Phys.* 90 (1983), 161; *J. Phys.* A16, 4177.

[10] N.E. MAVROMATOS (1988). *J. Phys.* A21, 2279.

Sigma models and Kac-Moody algebras

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1 Introduction

Symmetries and symmetry breaking play an important role in quantum field theory. In the beginning it were primarily finite-dimensional Lie groups and their Lie algebras which made their appearance in particle physics. With the advent of current algebras and dual models infinite-dimensional Lie algebras appeared on the stage. Current algebra is a quantum mechanical theory of elementary particles. However, one abstains here from the quantum fields of the particles. Instead of quantum fields one takes the operator fields of, for instance, currents and the energy-momentum tensor. Dual models were born out of the analysis of the scattering amplitudes of hadrons. Eventually this led to the string picture of hadrons and their vertex operators. The infinite-dimensional Lie algebra encountered in dual models is the Virasoro algebra. Current algebras are closely related to Kac-Moody algebras, another type of infinite-dimensional Lie algebras. In more recent times ideas similar to the vertex operators of the dual models were used in the representation theory of Kac-Moody algebras.

A central theme of this article are the sigma models. These models arose as a toy model for the description of a system of hadrons and their interactions: the sigma model of Gell-Mann and Lévy. Many aspects of current algebra are embodied in sigma models. Above all they are particularly attractive since they are Lagrangian field theories. Out of the sigma model of Gell-Mann and Lévy gradually developed the Wess-Zumino-Witten model. For this model are discussed its Kac-Moody algebra and its Bose-Fermi equivalence. Finally this model has also an application in the compactification of the bosonic string.

2 An easy route to Kac-Moody and Virasoro algebras

The purpose of the section is to indicate how Kac-Moody and Virasoro algebras arise in a simple differential geometric setting.

2.1 Loop groups

Let X be a finite-dimensional compact smooth manifold and let G be a finite-dimensional Lie group. The set of smooth maps $X \to G$ will be denoted by $\mathrm{Map}(X,G)$. It is now explained how the group structure of G gives rise to a group structure of $\mathrm{Map}(X,G)$. Let $\gamma_1: X \to G$ and $\gamma_2: X \to G$ be elements of $\mathrm{Map}(X,G)$. Then their product $\gamma_1 \cdot \gamma_2$ is defined by

$$(\gamma_1 \cdot \gamma_2)(x) = (\gamma_1(x)).(\gamma_2(x)) \tag{2.1.1}$$

for all x in X. This multiplication of elements of $\mathrm{Map}(X,G)$ is clearly associative. Furthermore, let e be the unit element of G, then the element ϵ of $\mathrm{Map}(X,G)$ defined by

$$\epsilon(x) = e \tag{2.1.2}$$

for all x in X is a unit element of the above defined multiplication in $\mathrm{Map}(X,G)$. For γ in $\mathrm{Map}(X,G)$ the element γ^{-1} of $\mathrm{Map}(X,G)$ is defined by

$$\gamma^{-1}(x) = (\gamma(x))^{-1} \tag{2.1.3}$$

and it is an inverse of γ. Hence, Map(X,G) equipped with the above multiplication is a group. In the particular case where X is a circle S^1 the group Map(S^1,G) is called a *loop group*. It is also denoted by $LG \equiv$ Map(S^1,G).

2.2 Lie algebra of a loop group

Let us in particular take for the circle S^1 the unit circle in the complex plane. And suppose for convenience sake that the Lie group G is a matrix group. The Lie algebra g of G then also consists of matrices. Let $\{T_a\}$ $(a = 1,..., r)$ be a basis of the Lie algebra g. It has the commutation relations

$$[T_a, T_b] = iC_{ab}^{\ c}\, T_c \tag{2.2.1}$$

where $C_{ab}^{\ c}$ are the structure constants and the summation convention is understood. Let γ be an element of the group LG and let z be a point on the unit circle S^1 in the complex plane, then $\gamma(z) \in G$ and for elements in a suitable neighbourhood of the unit element $e \in G$ one has

$$\gamma(z) = \exp(-i\theta^a(z)T_a) = 1 - i\theta^a(z)T_a + O(\theta^2) \tag{2.2.2}$$

where 1 denotes the unit matrix. The Laurent expansion of θ^a reads

$$\theta^a(z) = \sum_{n=-\infty}^{\infty} \theta^a_{-n} z^n \tag{2.2.3}$$

Inserting it into (2.2.2) gives

$$\gamma(z) = 1 - i \sum_{n,a} \theta^a_{-n} T^n_a + O\,(\theta^2) \tag{2.2.4}$$

where T^n_a is defined by

$$T^n_a = z^n T_a \quad (a = 1,..., r; n = 0, \pm 1, \pm 2,...) \tag{2.2.5}$$

From (2.2.4) it is clear that elements in a suitable neighbourhood of the unit element ϵ of LG have coordinates θ^a_{-n} $(a = 1,..., r; n = 0, \pm 1, \pm 2,...)$ and the matrices T^n_a are the corresponding generators. The Lie algebra lg of the loop group LG is spanned by $\{T^n_a\}_{n,a}$. It is called a *loop algebra* and it is clearly an infinite-dimensional Lie algebra. It can be taken to be a real or a complex Lie algebra. Here it will always be assumed to be a complex Lie algebra. The commutation relations of the generators T^n_a immediately follow from (2.2.1) and (2.2.5). They read

$$[T^m_a, T^n_b] = iC_{ab}^{\ c} T^{m+n}_c \tag{2.2.6}$$

Notice that

$$T^0_a = T_a \tag{2.2.7}$$

2.3 Central extension of a Lie algebra

We now introduce the so-called central extension L_c of a Lie algebra L by an abelian Lie algebra C. Firstly a central extension L_c is the direct sum of L and C:

$$L_c = L \oplus C \tag{2.3.1}$$

Let $\{T_a\}$ be a basis of L with the structure constants C_{ab}^c [see (2.2.1)]. Furthermore let $\{k_i\}$ $(i = 1,...,n)$ be a basis of the abelian Lie algebra C. The commutator $[T_a,T_b]$ in L_c has to be a linear combination of $\{T_a,k_i | a = 1,...,r; i = 1,...,n\}$. The vector space L_c is called a *central extension* of L by means of C when L_c is a Lie algebra with commutation relations

$$[T_a,T_b] = iC_{ab}^c T_c + id_{ab}^j k_j \tag{2.3.2}$$

$$[T_a,k_i] = 0 \tag{2.3.3}$$

$$[k_i,k_j] = 0 \tag{2.3.4}$$

Notice that in general (2.3.2) differs from (2.2.1). For finite-dimensional semisimple Lie algebras it can be shown (see section 1.3 of reference [1]) that there exists a basis $\{I_a,k_i | a = 1,...,r; 1 = 1,...,n\}$ where

$$I_a = T_a - \zeta_a^i k_i \tag{2.3.5}$$

with commutation relations

$$[I_a,I_b] = iC_{ab}^c I_c \tag{2.3.6}$$

$$[I_a,k_i] = 0 \tag{2.3.7}$$

$$[k_i,k_j] = 0 \tag{2.3.8}$$

Hence for finite-dimensional semisimple Lie algebras central extensions are essentially trivial since the commutation relations of the original algebra L are unchanged in this case. However, the central extensions of the loop algebra lg are in general not trivial. The non-trivial part of the central extension can however be proved to be one-dimensional (see again section 1.3 of reference [1]). The central extension lg_c of the loop algebra lg is a direct sum of lg and C, where C is a one-dimensional Lie algebra. Let C be spanned by k, then the commutation relations of the central extension of lg read

$$[T_a^m,T_b^n] = iC_{ab}^c T_c^{m+n} + m\delta_{ab}\delta_{m,-n}k \tag{2.3.9}$$

and

$$[T_a^m,k] = 0, \qquad [k,k] = 0 \tag{2.3.10}$$

Let a Cartan subalgebra h of the Lie algebra g of the finite-dimensional Lie group G be spanned by

$$H_p = c_p^a T_a \ (p = 1,...,N) \tag{2.3.11}$$

then $\{H_p,k | p = 1,...,N\}$ spans a Cartan subalgebra of lg_c [see also (2.2.7)]. From (2.3.9) - (2.3.11) follows

$$[H_p,T_b^n] = ic_p^a C_{ab}^c T_c^n \tag{2.3.12}$$

$$[k,T_b^n] = 0 \tag{2.3.13}$$

The coefficients in the right-hand sides of (2.3.12) and (2.3.13) do not depend on the index n. Because of this the roots of lg_c are degenerate. Indeed, let $\alpha = (\alpha_1, \ldots, \alpha_N)$ be a root of g, i.e., there exists an element E_α in g such that for $p = 1,...,N$

$$(\mathrm{ad}\, H_p)E_\alpha = [H_p, E_\alpha] = \alpha_p E_\alpha \tag{2.3.14}$$

where E_α is a linear combination of generators T_a:

$$E_\alpha = A^a_\alpha T_a \tag{2.3.15}$$

The degeneracy is now obvious, since

$$E^n_\alpha = z^n E_\alpha = A^a_\alpha T^n_a \tag{2.3.16}$$

also satisfies the eigenvalue equation

$$(\mathrm{ad}\, H_p)E^n_\alpha = \alpha_p E^n_\alpha \quad (p = 1,...,N) \tag{2.3.17}$$

for all $n = 0, \pm 1, \pm 2, \ldots$. In order to eliminate this degeneracy a further extension of the Lie algebra lg_c is performed in the next section.

2.4 Non-twisted affine Kac-Moody algebra

The so-called extension by a derivation of lg_c is defined to be the direct sum

$$g_{KM} = lg_c \oplus D \tag{2.4.1}$$

where D is a one-dimensional Lie algebra. Let D be spanned by an element d in D. The commutation relations of g_{KM} are taken to be

$$[T^m_a, T^n_b] = iC^c_{ab} T^{m+n}_c + m\delta_{ab}\delta_{m,-n} k \tag{2.4.2}$$

$$[T^m_a, k] = 0, \quad [k,k] = 0 \tag{2.4.3}$$

$$[d, T^n_a] = nT^n_a \tag{2.4.4}$$

$$[d,k] = 0 \quad [d,d] = 0 \tag{2.4.5}$$

The element d is an example of a so-called derivation. Notice that the commutation relations (2.4.2) and (2.4.3) of the subalgebra lg_c are not altered by this kind of extension. The complex Lie algebra g_{KM} is a so-called *non-twisted affine Kac-Moody algebra.* The subalgebra spanned by $\{H_p, k, d | p = 1,...,N\}$ is a Cartan subalgebra of g_{KM}. From (2.4.4) it is clear that its roots are no longer degenerate due to its extension by the derivation d. Notice that the dimension of the Cartan subalgebra is equal to $2+N$ where N is the dimension of the Cartan subalgebra of g.

2.5 Virasoro algebra

The Virasoro algebra is another example of an infinite-dimensional Lie algebra which is rather important in two-dimensional physical models e.g. string theory. It can also be obtained as the central extension of the Lie algebra of a group. Let us consider the set Diff(S^1) of all diffeomorphisms

$$\zeta : S^1 \rightarrow S^1 \tag{2.5.1}$$

of the unit circle S^1 in the complex plane. The product $\zeta_1.\zeta_2$ of the diffeomorphisms ζ_1 and ζ_2 from Diff(S^1) is defined by

$$(\zeta_1.\zeta_2)(z) = \zeta_1(\zeta_2(z)) \quad (z \in S^1) \tag{2.5.2}$$

With this multiplication Diff(S^1) is easily shown to be a group. The latter is also denoted by Diff(S^1). The unit element ϵ of Diff(S^1) is the identity map

$$\epsilon : z \in S^1 \rightarrow \epsilon(z) = z \in S^1 \tag{2.5.3}$$

The inverse element ζ^{-1} is the inverse of the map ζ.
Let H be the vector space of functions which are holomorphic on S^1. We now introduce a particular representation D of the above group Diff(S^1) with the vector space H as its representation space. Let f be a holomorphic function on S^1 then $D_\zeta : H \rightarrow H$ is defined for all ζ in Diff(S^1) by

$$(D_\zeta f)(z) = f(\zeta^{-1}(z)) \tag{2.5.4}$$

Note that D_ζ is a linear operator. These operators satisfy

$$D_{\zeta_1} D_{\zeta_2} = D_{\zeta_1.\zeta_2} \tag{2.5.5}$$

The map

$$D : \zeta \in \text{Diff}(S^1) \rightarrow D_\zeta \tag{2.5.6}$$

is a faithful representation of Diff(S^1). Elements in a suitable neighbourhood of the unit element of Diff(S^1) can be written as

$$\zeta(z) = z \exp(-i\epsilon(z)) = z - iz\epsilon(z) + O(\epsilon^2) \tag{2.5.7}$$

where ϵ is a function holomorphic on S^1. From (2.5.7) follows

$$\zeta^{-1}(z) = z + iz\epsilon(z) + O(\epsilon^2) \tag{2.5.8}$$

Using (2.5.4) one gets

$$(D_\zeta f)(z) = f(z) + iz\epsilon(z)\frac{d}{dz}f(z) + O(\epsilon^2) \tag{2.5.9}$$

Insertion of the Laurent expansion

$$\epsilon(z) = \sum_{n=-\infty}^{\infty} \epsilon_{-n} z^n \tag{2.5.10}$$

in (2.5.9) gives

$$(D_\zeta f)(z) = f(z) + i \sum_{n=-\infty}^{\infty} \epsilon_{-n} z^{n+1} \frac{d}{dz} f(z) + O(\epsilon^2) \tag{2.5.11}$$

The linear operators defined by

$$L_n = -z^{n+1}\frac{d}{dz} \tag{2.5.12}$$

on H span the Lie algebra of the group Diff(S^1). Their commutation relations, which follow immediately from (2.5.12), read

$$[L_n, L_m] = (m - n)L_{m+n} \tag{2.5.13}$$

The only non-trivial central extension of this Lie algebra is essentially an extension by a one-dimensional Lie algebra C. Let this C be spanned by c then its commutation relations can be shown to read (see section 1.3 of reference [2]):

$$[L_n, L_m] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m,-n} \tag{2.5.14}$$

$$[L_m, c] = 0 \tag{2.5.15}$$

The Lie algebra spanned by $\{L_m, c | m = 0, \pm 1, \pm 2, ...\}$ is called the *Virasoro algebra.*

3 Symmetries, current algebras and Schwinger terms

In this section the interrelation between symmetries and current algebras on the one hand and Kac-Moody algebras on the other is sketched. The physical formalism in which symmetries and current algebras are presented is quantum field theory. An entry to the latter theory is offered by classical Lagrangian field theory.

3.1 Lagrangian field theory

The ingredients of a classical Lagrangian field theory are the following:

- Spacetime. This is postulated to be an n-dimensional differentiable manifold M which is paracompact and Hausdorff. In the theory of special relativity spacetime is taken to be a four-dimensional Minkowski space. Although casual observation of the dimension of spacetime leads to $n = 4$ it is expedient for several reasons to consider here the case of an arbitrary positive integer n.
- Fields. These are sets $\{\phi_k | k = 1, \ldots, N\}$ of real- or complex-valued functions on spacetime M.
- Observables (measurable quantities). These are functions, usually polynomial, of the fields and their derivatives with respect to spacetime coordinates

$$F = F(\phi_1, \ldots, \phi_N, \partial_\mu \phi_1, \ldots, \partial_\mu \phi_N, ...) \tag{3.1.1}$$

The spacetime coordinates are denoted by

$$(x^\mu) = (x^0, x^1, \ldots, x^{n-1}) \quad (x^0 = ct) \tag{3.1.2}$$

wherein c is the speed of light. Some of the important observables are related to symmetry transformations [see section 3.3].

- Equations of motion. The evolution of the fields in the course of time is determined by the so-called *Euler-Lagrange equations*

$$\partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi_k)} - \frac{\partial L}{\partial \phi_k} = 0 \quad (k = 1, ..., N) \tag{3.1.3}$$

where L, called the *Lagrangian (density),* is a given function of the fields and their derivatives. Mostly L is a function of the fields and their first order derivatives:

$$L = L(\phi_k, \partial_\mu \phi_k) \tag{3.1.4}$$

This ensures that (3.1.3) is in general a second order partial differential equation. The Euler-Lagrange equations (3.1.3) are equivalent with Hamilton's action principle. This principle states that for all points x belonging to the interior of a region Ω in spacetime one has

$$\frac{\delta S}{\delta \phi_k(x)} = 0 \quad (k = 1,...,N) \tag{3.1.5}$$

In the left-hand side appears the variational derivative of the so-called *action* $S = S[\phi_1, \ldots, \phi_N]$. This is a functional of the fields, defined by

$$S = \int_\Omega L d^n x \quad (d^n x = dx^0 dx^1 \cdots dx^{n-1}) \tag{3.1.6}$$

The *canonical momentum vector* is defined by

$$\pi^{k\mu} = \frac{\partial L}{\partial(\partial_\mu \phi_k)} \tag{3.1.7}$$

Its zeroth component, denoted by

$$\pi^k = \frac{\partial L}{\partial(\partial_0 \phi_k)} \tag{3.1.8}$$

is called the *canonical momentum.* Assuming that all $\partial_0 \phi_k$ can be solved from (3.1.8) one has

$$\partial_0 \phi_k = f_k(\pi^1, \ldots, \pi^N, \phi_1, \ldots, \phi_N) \tag{3.1.9}$$

With this one defines the *Hamilton density*

$$H = H(\pi^1, \ldots, \phi_N) = \left[\sum_{k=1}^{N} \pi^k \dot{\phi}_k - L \right] \Big|_{\dot{\phi}_k = f_k(\pi,\phi)} \tag{3.1.10}$$

Its integral over spacetime is a functional of $\pi_1, \ldots, \pi_N, \phi_1, \ldots, \phi_N$ denoted by

$$H = \int H \, d^n x \tag{3.1.11}$$

and it is called the *Hamiltonian* of the system.
It is easily shown, using (3.1.5), that the Lagrangians L and

$$\tilde{L} := L + \partial_\mu G^\mu \tag{3.1.12}$$

where G^μ is a function of the fields and their derivatives give rise to the same equations of motion. Indeed Gauss' theorem implies

$$\tilde{S} = \int_\Omega \tilde{L} d^n x = \int_\Omega L d^n x + \int_\Omega \partial_\mu G^\mu \, d^n x \tag{3.1.13}$$
$$= S + \int_{\partial\Omega} G^\mu \, d\sigma_\mu$$

For x in the interior of Ω one has

$$\frac{\delta}{\delta\phi(x)} \int_{\partial\Omega} G^{\mu}\, d\sigma_{\mu} = 0 \tag{3.1.14}$$

since the integral only gets contributions from the boundary $\partial\Omega$ of Ω. From (3.1.13) and (3.1.14) follows then

$$\frac{\delta S}{\delta\phi_k(x)} = \frac{\delta \tilde{S}}{\delta\phi_k(x)} \tag{3.1.15}$$

Hence, according to Hamilton's action principle, both actions or both Lagrangians give rise to the same equation of motion.

3.2 Quantization

The quantum field theory corresponding to a classical Lagrangian field theory is characterized by the following features. The state of the system is described by a non-zero vector in a Hilbert space H. Observables are represented by self-adjoint operators on H. Each classical field $\phi_k = \phi_k(x)$ is replaced by a so-called quantum field on spacetime M. A quantum field on spacetime M is an operator-valued field on M, i.e. a map

$$\hat{\phi}_k : x \in M \rightarrow \hat{\phi}_k(x) \tag{3.2.1}$$

where the right-hand side is an operator on H for all x in M. Likewise the canonical momenta are taken to be operator fields

$$\hat{\pi}_k : x \in M \rightarrow \hat{\pi}_k(x) \tag{3.2.2}$$

where the right-hand since is an operator on H for all x in M, defined by [compare (3.1.8)]

$$\hat{\pi}_k(x) = \frac{\partial L}{\partial \dot{\phi}_k}\Big|_{\phi_k = \hat{\phi}_k} \tag{3.2.3}$$

The operator fields (3.2.1) and (3.2.3) are fundamental quantities in quantum field theory. They are sufficiently characterized by the following equal-time (anti-)commutation relations:

$$[\hat{\phi}_k(t,\mathrm{x}), \hat{\pi}^l(t,\mathrm{y})]_{-\epsilon} = i\hbar\delta(\mathrm{x}-\mathrm{y})\delta^l_k \tag{3.2.4}$$

$$[\hat{\phi}_k(t,\mathrm{x}), \hat{\phi}_l(t,\mathrm{y})]_{-\epsilon} = 0 = [\hat{\pi}^k(t,\mathrm{x}), \hat{\pi}^l(t,\mathrm{y})]_{-\epsilon} \tag{3.2.5}$$

where $\mathrm{x} = (x^1, \ldots, x^{n-1})$

$$[A,B]_{-\epsilon} = AB - \epsilon BA \quad (\epsilon = \pm 1) \tag{3.2.6}$$

and $\hbar = h/2\pi$ with h being Planck's constant. For $\epsilon = +1$ the fields are called *Bose fields* and for $\epsilon = -1$ *Fermi fields*. Although, for the sake of simplicity of notation, (3.2.4) and (3.2.5) are restricted to one type of such fields, systems containing both Fermi and Bose fields occur abundantly. In the latter case one mostly requires that all Bose fields (at time t) commute with all Fermi fields (at time t). From (3.2.4) it is obvious that quantum fields are rather singular

objects: they are actually operator-valued distributions.
The self-adjoint operators representing observables are obtained in the following way. The classical observable (3.1.1) is expressed in terms of fields ϕ_k, π^k and their spatial derivatives $\partial_m \phi_k, \partial_m \pi^k, \ldots (m=1, \ldots, n-1)$:

$$F = F(\phi_1, \ldots, \phi_N, \pi^1, \ldots, \pi^N, \partial_m \phi_1, \ldots, \partial_m \pi^1, \ldots) \tag{3.2.7}$$

The right-hand side is supposed to be a polynomial or at least a power series. The fields occurring in a term of this polynomial or power series can be chosen in an arbitrary order since they all commute, in particular

$$\phi(t,\mathbf{x})\pi(t,\mathbf{y}) = \pi(t,\mathbf{y})\phi(t,\mathbf{x}) \tag{3.2.8}$$

(indices are suppressed). The self-adjoint operator representing the observable corresponding to the classical observable (3.2.7) is obtained by performing the substitution

$$\phi_k \to \hat{\phi}_k, \ \pi^k \to \hat{\pi}^k \quad (k = 1, \ldots, N) \tag{3.2.9}$$

in the right-hand side of (3.2.7). The result of this substitution is in general not unique due to ordering ambiguities. Indeed, in the classical theory one has (3.2.8), whereas in the quantum mechanical theory

$$\hat{\phi}(t,\mathbf{x})\hat{\pi}(t,\mathbf{y}) \neq \hat{\pi}(t,\mathbf{y})\hat{\phi}(t,\mathbf{x}) \tag{3.2.10}$$

holds [see (3.2.9]. Clearly a procedure is needed which removes this ambiguity. An important example of such an ordering prescription is the so-called normal ordering.

3.3 Symmetry transformations and conserved currents

In this section we derive Noether's theorem, which states that to each symmetry transformation corresponds a conserved charge. We suppress the carets which indicate that we have to do with operators. Neglecting ordering ambiguities everything in this subsection applies both to classical an quantum field theory.
Instead of the field $\phi_k(x)$ we consider a one-parameter family of fields $\phi_k(x,\epsilon) (\epsilon \in \mathbb{R})$ such that in particular $\phi_k(x,0) = \phi_k(x)$. We assume that

$$\phi_k(x,\epsilon) = \phi_k(x) + \epsilon F(\phi_k, \partial_\mu \phi_k) + O(\epsilon^2) \tag{3.3.1}$$

for $\epsilon \to 0$. For convenience sake from now on the index k of ϕ_k will be suppressed. A transformation

$$\phi(x) \to \phi(x,\epsilon) \tag{3.3.2}$$

is called a *symmetry transformation* if, without the use of the equations of motion (3.1.3),

$$\frac{d}{d\epsilon} L(\phi(x,\epsilon), \partial_\mu \phi(x,\epsilon))|_{\epsilon=0} = \partial_\mu \Lambda^\mu \tag{3.3.3}$$

holds where Λ^μ is a function of the fields and their derivatives. For a symmetry transformation one has

$$L(\phi(x,\epsilon),\partial_\mu\phi(x,\epsilon)) = L(\phi(x),\partial_\mu\phi(x)) + \epsilon\partial_\mu\Lambda^\mu + O(\epsilon^2) \tag{3.3.4}$$

for $\epsilon \to 0$. Hence both Lagrangians differ by a divergence and in view of the statement following (3.1.12) the fields $\phi_k(x,\epsilon)$ and $\phi_k(x)$ then satisfy the same equations of motion up to terms of $O(\epsilon^2)$ for $\epsilon \to 0$. One the other hand, using the equations of motion (3.1.3), one actually arrives at an equation which has the form of (3.3.3). Namely, using the chain rule and (3.3.1), one gets

$$\frac{d}{d\epsilon}L(\phi(x,\epsilon),\partial_\mu\phi(x,\epsilon))|_{\epsilon=0} = \frac{\partial L}{\partial\phi}F + \frac{\partial L}{\partial(\partial_\mu\phi)}\partial_\mu F \tag{3.3.5}$$

$$= (\partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi)})F + \frac{\partial L}{\partial(\partial_\mu\phi)}\partial_\mu F = \partial_\mu(\frac{\partial L}{\partial(\partial_\mu\phi)}F)$$

From (3.3.3) and (3.3.5) follows

$$\partial_\mu(\frac{\partial L}{\partial(\partial_\mu\phi)}F - \Lambda^\mu) = 0 \tag{3.3.6}$$

This is called a continuity equation or differential form of a conservation law. Defining a so-called *Noether current* by

$$J^\mu = \frac{\partial L}{\partial(\partial_\mu\phi)}F - \Lambda^\mu \tag{3.3.7}$$

(3.3.6) reads

$$\partial_\mu J^\mu = 0 \tag{3.3.8}$$

A current satisfying (3.3.8) is called a *conserved current.* The spatial integral of the zeroth component of a conserved current is time-independent when the current falls off sufficiently fast at spatial infinity. Indeed let

$$Q(t) := \int J^0(x)d\mathbf{x} \tag{3.3.9}$$

where $x = (ct,x^1,\ldots,x^{n-1})$ and $d\mathbf{x} = dx^1dx^2\cdots dx^{n-1}$. Quantities of the type (3.3.9) are called *charges.* One has, using Gauss' theorem,

$$\frac{dQ}{dt} = x\int\partial_0 J^0 d\mathbf{x} = -c\int\nabla.\mathbf{J}d\mathbf{x} \tag{3.3.10}$$

$$= -c\lim_{R\to\infty}\int_{B(R)}\nabla.\mathbf{J}\,d\mathbf{x} = -\lim_{R\to\infty}\int_{\partial B(R)}\mathbf{J}.d\mathbf{S}$$

where $\nabla = (\partial_1,\ldots,\partial_{n-1}), \mathbf{J} = (J^1,\ldots,J^{n-1})$ and $B(R)$ a ball with radius R. When $|\mathbf{J}|$ approaches zero sufficiently fast at spatial infinity, the surface integral over the sphere also goes to zero for $R\to\infty$. Hence

$$\frac{dQ}{dt} = 0 \tag{3.3.11}$$

A time-independent charge is also called a *conserved charge* and (3.3.11) is called a *(global) conservation law.*

Some quantities in physics are almost conserved. These fit nicely into the following slight generalization of the above results. Take for a one-parameter family of fields again (3.3.1). Suppose, however, that instead of (3.3.3) one has the more general result

$$\frac{d}{d\epsilon}L(\phi(x,\epsilon),\partial_\mu\phi(x,\epsilon))|_{\epsilon=0} = \partial_\mu\Lambda^\mu + \Delta \tag{3.3.12}$$

where Λ^μ and Δ are functions of the fields ϕ and their derivatives. The sum in the right-hand side is defined by the left-hand side and its decomposition into a divergence $\partial_\mu\Lambda^\mu$ and Δ is certainly not unique. From (3.3.5) and (3.3.12) follows

$$\partial_\mu(\frac{\partial L}{\partial(\partial_\mu\phi)}F - \Lambda^\mu) = \Delta \tag{3.3.13}$$

or [see (3.3.7)]

$$\partial_\mu J^\mu = \Delta \tag{3.3.14}$$

This is called a *balance law.* Thus we have to do with a so-called *partial conservation law* when there exists a decomposition (3.3.12) such that Δ can be considered to be small.

For convenience sake only a one-parameter family of fields was considered above. For an r-parameter family

$$\phi_k(x,\epsilon^1, \ldots ,\epsilon^r) = \phi_k(x) + \epsilon^i F_i(\phi_k,\partial_\mu\phi_k) + O(\epsilon^2) \tag{3.3.15}$$

one has, without using the equations of motion,

$$\frac{\partial}{\partial\epsilon^i}L(\phi(x,\epsilon^1,...),\ \partial_\mu\phi(x,\epsilon^1,...))|_{\epsilon=0} = \partial_\mu\Lambda_i^\mu + \Delta_i \tag{3.3.16}$$

Using on the other hand the equations of motion one has [compare (3.3.5)]

$$\frac{\partial}{\partial\epsilon^i}L(\phi(x,\epsilon^1,...),\partial_\mu\phi(x,\epsilon^1, \ldots ,)|_{\epsilon=0} = \partial_\mu(\frac{\partial L}{\partial(\partial_\mu\phi)}F_i) \tag{3.3.17}$$

Defining [compare (3.3.7)]

$$J_i^\mu = \frac{\partial L}{\partial(\partial_\mu\phi)}F_i - \Lambda_i^\mu \tag{3.3.18}$$

one gets

$$\partial_\mu J_i^\mu = \Delta_i \tag{3.3.19}$$

For a symmetry transformation (3.3.15) there exists by definition a choice of the decomposition (3.3.16) such that $\Delta_i = 0\,(i = 1, \ldots ,r)$ and one then has

$$\partial_\mu J_i^\mu = 0 \tag{3.3.20}$$

This subsection is finished by giving two examples of conserved Noether currents. The first is the canonical energy-momentum tensor and its corresponding symmetry transformations are translations in spacetime. The second are the charges corresponding to internal symmetry transformations. A translation in Minkowski spacetime

$$x^\kappa \to x^\kappa + \epsilon^\kappa \tag{3.3.21}$$

gives rise to the following transformation of fields

$$\phi(x,\epsilon) = \phi(x+\epsilon) = \phi(x) + \epsilon^\kappa \partial_\kappa \phi + O(\epsilon^2) \tag{3.3.22}$$

This is an example of (3.3.15). In the case that L does not depend explicitly on x, i.e. L depends on x only via the fields and their derivatives, one has, without using the equations of motion [compare (3.3.16)],

$$\begin{aligned} &\frac{\partial}{\partial \epsilon^\kappa} L(\phi(x,\epsilon), \partial_\mu \phi(x,\epsilon))|_{\epsilon=0} = \\ &\frac{\partial}{\partial \epsilon^\kappa} L(\phi(x+\epsilon), \partial_\mu \phi(x+\epsilon))|_{\epsilon=0} = \\ &\frac{\partial}{\partial x^\kappa} L(\phi(x), \partial_\mu \phi(x)) = \partial_\mu(\delta^\mu_\kappa L) \end{aligned} \tag{3.3.23}$$

Comparison of (3.3.22) and (3.3.23) with (3.3.15) and (3.3.16) gives

$$F_\kappa = \partial_\kappa \phi, \quad \Lambda^\mu_\kappa = \delta^\mu_\kappa \quad L, \Delta_\kappa = 0 \tag{3.3.24}$$

The current (3.3.18) becomes in this case

$$\Theta^\mu_\kappa := \frac{\partial L}{\partial(\partial_\mu \phi)} \partial_\kappa \phi - \delta^\mu_\kappa L \tag{3.3.25}$$

It is conserved, i.e.

$$\partial_\mu \Theta^\mu_\kappa = 0 \tag{3.3.26}$$

when L does not depend explicitly on x. The current Θ^μ_κ is called the *canonical energy-momentum tensor*. Notice that [see (3.1.10)]

$$\Theta^0_0 = H \tag{3.3.27}$$

Thus the Hamiltonian [see (3.1.11)] is given by

$$H = \int \Theta^0_0 d\mathbf{x} \tag{3.3.28}$$

More generally one defines

$$P^\mu = \frac{1}{c} \int \Theta^{0\mu} d\mathbf{x} \tag{3.3.29}$$

This is called the *relativistic momentum*, and it is also denoted by

$$P^\mu \equiv \left(\frac{H}{c}, \mathbf{P}\right) \tag{3.3.30}$$

Because of (3.3.26) it is conserved:

$$\frac{dP^\mu}{dt} = 0 \tag{3.3.31}$$

This is the law of conservation of energy and momentum.
The transformations (3.3.15) which appear in the second example leave the spacetime coordinates alone. Let us suppose that the fields ϕ_k are collected in a column vector ϕ. Then we consider now transformations having the form

$$\phi(x,\epsilon) = U(\epsilon^1, \ldots, \epsilon^r)\phi(x) \tag{3.3.32}$$

where U is a unitary matrix representation of a Lie group G. Furthermore these transformations are assumed to leave the Lagrangian invariant. That is

$$L(\phi(x,\epsilon), \partial_\mu\phi(x,\epsilon)) = L(\phi(x), \partial_\mu\phi(x)) \tag{3.3.33}$$

holds for group elements in a neighbourhood of the identity element of G. Such transformations are called *internal symmetry transformations.* An example of a Lagrangian which has an invariance (3.3.33) is

$$L = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi \tag{3.3.34}$$

It is clearly invariant under all unitary transformations (3.3.32). Comparison of (3.3.16) and (3.3.33) shows that one can choose

$$\Lambda_a^\mu = 0, \quad \Delta_a = 0 \tag{3.3.35}$$

Hence (3.3.32) is a symmetry transformation for the Lagrangian (3.3.34). Let

$$U(\epsilon^1, \ldots, \epsilon^r) = 1 + i\epsilon^a T_a + O(\epsilon^2) \tag{3.3.36}$$

for $\epsilon^a \to 0$, then

$$\phi(x,\epsilon) = \phi(x) + i\epsilon^a T_a\phi(x) + O(\epsilon^2) \tag{3.3.37}$$

or [compare (3.3.15)]

$$F_a = iT_a\phi \tag{3.3.38}$$

The matrices T_a are hermitian since U is unitary [see (3.3.36)] and they have commutation relations (2.2.1). The corresponding conserved current is given by [see (3.3.18)]

$$J_a^\mu = -\pi^\mu iT_a\phi \tag{3.3.39}$$

where the additional minus sign is conventional. This current satisfies of course

$$\partial_\mu J_a^\mu = 0 \tag{3.3.40}$$

The zeroth component is called *charge density,* and

$$Q_a = \int J_a^0 d\mathbf{x} \tag{3.3.41}$$

is called a *charge.*

3.4 Current algebras and Schwinger terms

In quantum field theory the operator in the right-hand side of (3.3.39) has to be normal ordered in order to become a well-defined quantity. In the calculation of the equal time commutation relation of charge densities this can be ignored since normal ordering only gives an (infinite) c-number. The equal time commutation relation is easily calculated by means of the (anti-) commutation relations (3.2.4) - (3.2.6). One obtains

$$[J_a^0(t,\mathrm{x}),J_b^0(t,\mathrm{y})]= \tag{3.4.1}$$

$$\pi^k(t,\mathrm{x})i(T_a)_k^l\phi_l(t,\mathrm{x})\pi^m(t,\mathrm{y})i(T_b)_m^n\phi_n(t,\mathrm{y})$$

$$-\pi^m(t,\mathrm{y})i(T_b)_m^n\phi_n(t,\mathrm{y})\pi^k(t,\mathrm{x})i(T_a)_k^l\phi_l(t,\mathrm{x})=$$

$$\pi^k(t,\mathrm{x})i(T_a)_k^l[\epsilon\pi^m(t,\mathrm{y})\phi_l(t,\mathrm{x}) + i\hbar\delta(\mathrm{x}-\mathrm{y})\delta_l^m]i(T_b)_m^n\phi_n(t,\mathrm{y})$$

$$- \pi^m(t,\mathrm{y})i(T_b)_m^n[\epsilon\pi^k(t,\mathrm{x})\phi_n(t,\mathrm{y}) + i\hbar\delta(\mathrm{x}-\mathrm{y})\delta_n^k]i(T_a)_k^l\phi_l(t,\mathrm{x})$$

$$= -\epsilon\pi^k(t,\mathrm{x})\pi^m(t,\mathrm{y})\phi_l(t,\mathrm{x})\phi_n(t,\mathrm{y})(T_a)_k^l(T_b)_m^n$$

$$+\epsilon\pi^k(t,\mathrm{x})\pi^m(t,\mathrm{y})\phi_l(t,\mathrm{x})\phi_n(t,\mathrm{y})(T_a)_a^l(T_b)_m^n$$

$$- i\hbar\delta(\mathrm{x}-\mathrm{y})\{\pi^k(t,\mathrm{x})(T_aT_b)_k^n\phi_n(t,\mathrm{x}) + \pi^m(t,\mathrm{x})(T_bT_a)_m^l\phi_l(t,\mathrm{x})\}$$

$$= -\pi^k(t,\mathrm{x})([T_a,T_b])_k^n\phi_n(t,\mathrm{x})i\hbar\delta(\mathrm{x}-\mathrm{y})$$

By means of (2.2.1) and definition (3.3.39) this gives

$$[J_a^0(t,\mathrm{x}),J_b^0(t,\mathrm{y})] = i\hbar\delta(\mathrm{x}-\mathrm{y})C_{ab}^{\ \ c}J_c^0(t,\mathrm{x}) \tag{3.4.2}$$

Integration of both charge densities in the left-hand side of (3.4.2) over x and y respectively gives [see (3.3.41)]

$$[Q_a,Q_b] = iC_{ab}^{\ \ c}Q_c \tag{3.4.3}$$

where we have chosen natural units $h/2\pi=c=1$. Hence the charges are the generators of a Lie algebra. This algebra is isomorphic to the Lie algebra of the generators T_a[compare (2.2.1)]. The latter is the Lie algebra of the Lie group of symmetry transformations (3.3.32). Another consequence of (3.4.2) reads

$$[Q_a,J_b^0(t,\mathrm{y})] = iC_{ab}^{\ \ c}J_c^0(t,\mathrm{y}) \tag{3.4.4}$$

This is obtained by integrating (3.4.2) with respect to x. Under Lorentz transformations Q_a is a scalar operator and J_b^μ a vector operator field. Hence, it follows from (3.4.4) that for the spatial components of the current operator one has

$$[Q_a,J_b^i(t,\mathrm{y})] = iC_{ab}^{\ \ c}J_c^i(t,\mathrm{y}) \tag{3.4.5}$$

We now turn to the calculation of the commutator

$$[J_a^0(t,\mathrm{x}),J_b^i(t,\mathrm{y})] \tag{3.4.6}$$

Once the answer is known, integration of the charge density in this commutator has to give back (3.4.5). One might actually guess the answer. However, at this point one has to be careful. In view of this we first discuss a crucial property of quantum field theories: microcausality. Microcausality springs from a less severe restriction which is sometimes called Einstein causality. From quantum mechanics it is known that observables which can be measured simultaneously are represented by commuting self-adjoint operators and vice versa. The corresponding measurements are called compatible. In the theory of special relativity one comes to the conclusion that spacelike separated events cannot influence one another. Taken together this leads to the following assertion. Let Ω_1 and Ω_2 be two spacelike separated regions in Minkowski space-time. This means that for all x in Ω_1 and all y in Ω_2 the events x and y are spacelike separated, i.e.

$$(x^0-y^0)^2 - (\mathrm{x}-\mathrm{y})^2 < 0 \tag{3.4.7}$$

Let $A(\Omega_1)$ be a local observable restricted to the region Ω_1 and $B(\Omega_2)$ a local observable restricted to Ω_2. Since the spacelike separated regions Ω_1 and Ω_2 cannot influence one another, the measurements of $A(\Omega_1)$ and $B(\Omega_2)$ are compatible. Hence

$$[A(\Omega_1), B(\Omega_2)] = 0 \tag{3.4.8}$$

for Ω_1 and Ω_2 spacelike separated regions. This property is called Einstien causality. It leads in particular to

$$[A(x), B(y)] = 0 \tag{3.4.9}$$

for x and y space like separated events.
The fields $A(x)$ and $B(y)$ can be thought of as observables like a current, charge density or energy density. Einstein causality is implied by the stronger requirement [see (3.2.6)]

$$[\phi_k(x), \phi_l(y)]_{-\epsilon} = 0 \tag{3.4.10}$$

for all spacelike separated events x and y. This postulate is called microcausality.
For Bose fields ($\epsilon = +1$) one immediately sees that microcausality implies Einstein causality since $A(\Omega_1)$ then only consists of fields $\phi_k(x)$ and their derivatives with x belonging to Ω_1 and $B(\Omega_2)$ consists of fields $\phi_k(y)$ and their derivatives with y belonging to Ω_2. From (3.4.10) follows that all fields $\phi_k(x)$ and their derivatives commute with all fields $\phi_k(y)$ and their derivatives for spacelike separated events x and y. Hence (3.4.9) follows. To show the analogous statement for Fermi fields one can use the relation

$$[AB, C] = A[B, C]_+ - [A, C]_+ B \tag{3.4.11}$$

thereby expressing a commutator of local observables in terms of anticommutators of Fermi fields.
Microcausality determines the commutator between time-components and space-components of currents to a large extend. It says that

$$[J_a^0(t, \mathrm{x}), J_b^i(t, \mathrm{y})] = 0 \tag{3.4.12}$$

for $\mathrm{x} \neq \mathrm{y}$. Hence this commutator is a distribution with support $\mathrm{x} = \mathrm{y}$. But such a distribution is a finite linear combination of $\delta(\mathrm{x}-\mathrm{y})$ and its spatial derivatives. Consequently

$$[J_a^0(t,\mathbf{x}),J_b^i(t,\mathbf{y})] = iC_{ab}^c J_c^i(t,\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) + S_{ab}^{ij}\,\partial_j\delta(\mathbf{x}-\mathbf{y}) + \cdots \tag{3.4.13}$$

The first term in the right-hand side of (3.4.13) is fixed by the requirement that integration of (3.4.13) with respect to x has to give (3.4.5). The terms in (3.4.13) containing spatial derivatives are called Schwinger terms. Their integrals with respect to x are equal zero. Of course they could all be zero themselves. We now show that this is not true in general. Let us consider a conserved current J^μ i.e.

$$\partial_0 J^0 + \partial_i J^i = 0 \tag{3.4.14}$$

with a Schwinger term C^i defined by

$$[J^0(t,\mathbf{x}),J^i(t,\mathbf{y})] = iJ^i(t,\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) + C^i(\mathbf{x},\mathbf{y}) \tag{3.4.15}$$

and vacuum expectation value equal to zero:

$$\langle 0|J^\mu|0\rangle = 0 \tag{3.4.16}$$

From (3.4.15) and (3.4.16) follows

$$\langle 0|C^i(\mathbf{x},\mathbf{y})|0\rangle = \langle 0|[J^0(t,\mathbf{x}),J^i(t,\mathbf{y})]|0\rangle \tag{3.4.17}$$

Hence, using (3.4.14),

$$\langle 0|\frac{\partial}{\partial y^i}C^i(\mathbf{x},\mathbf{y})|0\rangle = \langle 0|[J^0(t,\mathbf{x}),\frac{\partial}{\partial y^i}J^i(t,\mathbf{y})]|0\rangle = -\langle 0|[J^0(t,\mathbf{x}),\frac{\partial}{\partial t}J^0(t,\mathbf{y})]|0\rangle \tag{3.4.18}$$

Application of the Heisenberg equation, which reads

$$i\frac{\partial}{\partial t}A(t,\mathbf{y}) = [A(t,\mathbf{y}),H] \tag{3.4.19}$$

gives

$$\langle 0|\frac{\partial}{\partial y^i}C^i(\mathbf{x},\mathbf{y})|0\rangle = i\langle 0|[J^0(t,\mathbf{x}),[J^0(t,\mathbf{y}),H]]|0\rangle = -\langle 0|J^0(t,\mathbf{x})HJ^0(t,\mathbf{y}) + J^0(t,\mathbf{y})HJ^0(t,\mathbf{x})|0\rangle \tag{3.4.20}$$

since the energy of the vacuum is zero:

$$H|0\rangle = 0 = \langle 0|H \tag{3.4.21}$$

Let $f=f(\mathbf{x})$ be a real-valued function which vanishes for $|\mathbf{x}|\to\infty$. Then partial integration gives

$$-i\int\int d\mathbf{x}d\mathbf{y}\langle 0|C^i(\mathbf{x},\mathbf{y})|0\rangle f(\mathbf{x})\frac{\partial}{\partial y^i}f(\mathbf{y}) = \tag{3.4.22}$$

$$i\int\int d\mathbf{x}d\mathbf{y}\langle 0|\frac{\partial}{\partial y^i}C^i(\mathbf{x},\mathbf{y})|0\rangle f(\mathbf{x})f(\mathbf{y}) =$$

Defining

$$F = \int J^0(t, \mathrm{x}) f(\mathrm{x}) d\,\mathrm{x} \tag{3.4.23}$$

and using (3.4.20) gives

$$i \int\int d\,\mathrm{x} d\,\mathrm{y} \langle 0|C^i(\mathrm{x},\mathrm{y})|0\rangle f(\mathrm{x}) \frac{\partial}{\partial y^i} f(\mathrm{y}) = 2\langle 0|FHF|0\rangle \tag{3.4.24}$$

From the hermiticity of J^0 and the real-valuedness of f follows that F is hermitian. Finally, we show that

$$\langle 0|FHF|0\rangle > 0 \tag{3.4.25}$$

Taking $J^0(t, \mathrm{x}) \neq 0$, not all matrix elements $<0|F|n>$ can be equal to zero. Notice however that $<0|F|0>=0$ [see (3.4.16)]. Furthermore, we assume that the energy eigenvalues E_n of the Hamiltonian H are non-negative. Denoting the corresponding normalized energy eigenvectors by $|n>$ one has

$$H|n\rangle = E_n|n\rangle \tag{3.4.26}$$

$$\langle m|n\rangle = \delta_{mn} \tag{3.4.27}$$

and

$$\sum_n |n\rangle\langle n| = 1 \tag{3.4.28}$$

where 1 stands for the unit operator one the Hilbert space of the state vectors. Only the vacuum state, denoted by $|0>$, has energy eigenvalue $E_0 = 0$. The proof of (3.4.25) runs as follows:

$$\begin{aligned} \langle 0|FHF|0\rangle &= \sum_{m,n} \langle 0|F|m\rangle\langle m|H|n\rangle\langle n|F|0\rangle \\ &= \sum_{m,n} \langle 0|F|m\rangle E_n \delta_{mn} \langle n|F|0\rangle \\ &= \sum_{n\neq 0} \langle 0|F|n\rangle E_n \langle n|F|0\rangle \end{aligned} \tag{3.4.29}$$

The hermiticity of F implies

$$\langle 0|F|n\rangle^* = \langle n|F|0\rangle \tag{3.4.30}$$

Hence

$$\langle 0|FHF|0\rangle = \sum_{n\neq 0} |\langle 0|F|n\rangle|^2 E_n > 0 \tag{3.4.31}$$

since all E_n occurring in the right-hand side are positive and not all $|<0|F|n>|^2$ are zero. This implies indeed that the right-hand side of (3.4.31) is in general positive. Hence (3.4.25) holds and it asserts that the right-hand side of (3.4.24) is unequal zero. Hence the Schwinger term C_i appearing in its left-hand side has to be unequal zero.

Thus from general assumptions like microcausality, Lorentz invariance and positivity of the energy spectrum one finds that Schwinger terms are in general unequal zero. In calculations based on a formal application of canonical commutation relations Schwinger terms are however often lost.

3.5 Current algebra in two-dimensional spacetime and its Kac-Moody algebra

We consider the untwisted affine Kac-Moody algebra (2.4.1) for a compact finite-dimensional Lie algebra g. For the present only the commutation relations (2.4.2) and (2.4.3) are of any importance. Let x be a real number then we define a current $J_a = J_a(x)$ by

$$J_a(x) = \frac{\hbar}{L} \sum_{n=-\infty}^{\infty} T_a^{-n} \exp(2\pi inx/L) \tag{3.5.1}$$

This current is actually defined on a circle since

$$J_a(x) = J_a(x+L) \tag{3.5.2}$$

We now calculate the commutation relations of these current using those of the generators of the Kac-Moody algebra. One has

$$[J_a(x), J_b(0)] = \left(\frac{\hbar}{L}\right)^2 \left[\sum_{n=-\infty}^{\infty} T_a^{-n} \exp(2\pi inx/L), \sum_{m=-\infty}^{\infty} T_b^{-m}\right] \tag{3.5.3}$$

$$= \left(\frac{\hbar}{L}\right)^2 \sum_{n,m} \exp(2\pi inx/L)[T_a^{-n}, T_b^{-m}]$$

Insertion of (2.4.2) gives

$$[J_a(x), J_b(0)] = \tag{3.5.4}$$

$$\left(\frac{\hbar}{L}\right)^2 \sum_m \exp(2\pi imx/L) \sum_l \exp(2\pi ilx/L) T_c^{-l} iC_{ab}^c$$

$$- \left(\frac{\hbar}{L}\right)^2 k\delta_{ab} \sum_{n,m} \exp(2\pi inx/L)\delta_{n,-n} m$$

$$= \left(\frac{\hbar}{L}\right) \sum_m \exp(2\pi imx/L) J_c(x) iC_{ab}^c$$

$$- \left(\frac{\hbar}{L}\right)^2 k\delta_{ab} \sum_m \exp(2\pi imx/L) m$$

By differentiating Poisson's summation formula

$$\frac{1}{L} \sum_m \exp(-2\pi imx/L) = \sum_m \delta(x+mL) = \delta(x) \tag{3.5.5}$$

for x in the interval $(0,L)$ one gets

$$\frac{-2\pi i}{L^2} \sum_m \exp(-2\pi imx/L) m = \delta'(x) \tag{3.5.6}$$

Insertion of (3.5.5) and (3.5.6) into (3.5.4) gives

$$[J_a(x),J_b(0)] = i\hbar C_{ab}^{c} J_c(0)\delta(x) + \frac{i\hbar^2}{2\pi} k\delta_{ab}\delta'(x) \tag{3.5.7}$$

Hence the current (3.5.1) satisfies a current algebra commutation relation with a Schwinger term [compare (3.4.13)]. In the present case the Schwinger term of the current algebra corresponds to the central extension of the loop algebra.

4 Sigma model of Gell-Mann and Lévy

The sigma model of Gell-Mann and Lévy is a charming toy model for a system of nucleons (protons and neutrons) and pions in interaction. Moreover it gives a nice example of spontaneous symmetry breaking. The pions appear as Goldstone bosons associated with this spontaneous symmetry breaking. The purpose it serves here is that of a stepping stone towards the non-linear sigma models.

4.1 Building the Lagrangian.

Input in the construction of the Lagrangian of the σ-model is the free Lagrangian L_0 of massless protons and neutrons. Protons and neutrons are spin-1/2-particles and they are described by Dirac fields. The Lagrangian L_D of a Dirac field q with mass m reads

$$L_D = \bar{q}(i\gamma^\mu \partial_\mu - m)q \tag{4.1.1}$$

where $q = q(x)$ is a complex-valued four-component column vector, the unitary 4×4-matrices γ^μ are the so-called Dirac matrices, characterized by the anti-commutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{4.1.2}$$

where $(\eta^{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$ and

$$\bar{q} := q^\dagger \gamma^0 \tag{4.1.3}$$

Denoting the Dirac field of the proton by p and that of the neutron by n the input Lagrangian L_0 of the model reads

$$L_0 = \bar{p} i\gamma^\mu \partial_\mu p + \bar{n} i\gamma^\mu \partial_\mu n \tag{4.1.4}$$

Both Dirac fields can be collected into an eight-component column vector

$$\psi = \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{4.1.5}$$

Then

$$L_0 = \bar{\psi} i\gamma^\mu \partial_\mu \psi \tag{4.1.6}$$

with

$$\bar{\psi} = \psi^+ (\gamma^0 \otimes 1) \equiv \psi^+ \gamma^0 \tag{4.1.7}$$

and where 1 is the 2×2 unit matrix. The latter will often be suppressed. The Lagrangian (4.1.6) is obviously invariant under transformations

$$\psi \rightarrow \hat{\psi} \equiv \begin{bmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \end{bmatrix} = U \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{4.1.8}$$

where U is a unitary 2×2-matrix. Hence the group of unitary 2×2-matrices is the symmetry group of the Lagrangian L_0. Each unitary 2×2-matrix U can be factorized as $U = (\det U)V$ where $\det U$ is a phase factor and V is a unitary 2×2-matrix with determinant equal to one. Hence the symmetry group of the Lagrangian is $U(1) \times SU(2)$. We will concentrate on the $SU(2)$ symmetry. The Lagrangian L_0 harbors an even larger symmetry due to the fact that the nucleons are taken to be massless. In order to show this we introduce the so-called left- and right-handed parts of a Dirac field, defined by

$$\psi_L = \frac{1-\gamma_5}{2}\psi, \quad \psi_R = \frac{1+\gamma_5}{2}\psi \tag{4.1.9}$$

where

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{4.1.10}$$

Clearly one has

$$\psi = \psi_L + \psi_R \tag{4.1.11}$$

The operators

$$P_L = \frac{1}{2}(1-\gamma_5), \quad P_R = \frac{1}{2}(1+\gamma_5) \tag{4.1.12}$$

are projection operators since they are hermitian and idempotent:

$$P_L^2 = P_L, \quad P_R^2 = P_R \tag{4.1.13}$$

The free massless Lagrangian (4.1.6) can be decomposed into the left- and right-handed fields since

$$\begin{aligned} L_0 &= \bar{\psi} i\gamma^\mu \partial_\mu \psi_L + \bar{\psi} i\gamma^\mu \partial_\mu \psi_R = \\ &\bar{\psi} i\gamma^\mu \partial_\mu P_L \psi_L + \bar{\psi} i\gamma^\mu \partial_\mu P_R \psi_R = \\ &\bar{\psi} P_R i\gamma^\mu \partial_\mu \psi_L + \bar{\psi} P_L i\gamma^\mu \partial_\mu \psi_R \end{aligned} \tag{4.1.14}$$

Hence

$$L_0 = \bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R \tag{4.1.15}$$

A mass-term gives rise to cross-terms between left- and right-handed fields since

$$m\bar{\psi}\psi = m(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R) \tag{4.1.16}$$

[see (4.1.26) below] and a decomposition similar to (4.1.15) is out of the question. Instead of the symmetry transformations (4.1.8) one perceives via (4.1.15) a larger symmetry group of L_0 namely

$$\psi_L \rightarrow \hat{\psi}_L = U_L \psi_L, \quad \psi_R \rightarrow \hat{\psi}_R = U_R \psi_R \tag{4.1.17}$$

where U_L and U_R are arbitrary unitary 2×2-matrices. Hence the symmetry group of L_0 is $U(1)_L \times SU(2)_L \times U(1)_R \times SU(2)_R$. We will concentrate on its subgroup $SU(2)_L \times SU(2)_R$. Symmetries which discriminate between the left- and right-handedness are coined *chiral.*

Next we add to L_0 a Lagrangian of the pions and the interaction Lagrangian of nucleons and pions. The pions are described by spinless fields. The interaction between the nucleons and the pions is taken to be a so-called Yukawa coupling, that is an interaction Lagrangian sesquilinear in the nucleon field and linear in the pion field. Interaction Lagrangians of this kind which are $SU(2)_L \times SU(2)_R$-invariant are easily seen to arise when the pions are represented by a 2×2-matrix of spinless fields $\Sigma = \Sigma(x)$ transforming under $SU(2)_L \times SU(2)_R$ as

$$\Sigma \rightarrow \hat{\Sigma} = U_L \Sigma U_R^{\dagger} \tag{4.1.18}$$

An immediate consequence of (4.1.17) and (4.1.18) is the invariance under $SU(2)_L \times SU(2)_R$ of the Yukawa interaction Lagrangian

$$L_Y = -g\bar{\psi}_L \Sigma \psi_R - g\bar{\psi}_R \Sigma^{\dagger} \psi_L \tag{4.1.19}$$

The terms in the right-hand side of (4.1.19) are each others complex-conjugate and consequently L_Y is real. The spinless fields contained in Σ can be made explicit by introducing a basis for the 2×2-matrices. Let us take for the basis of the 2×2-matrices $\{1, \tau^1, \tau^2, \tau^3\}$ where τ^i $(i = 1,2,3)$ are the Pauli matrices

$$\tau^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.1.20}$$

Then Σ can be decomposed as

$$\Sigma(x) = \sigma(x)1 + i\tau^a \pi_a(x) \tag{4.1.21}$$

where σ and $\pi_a (a = 1,2,3)$ are chosen to be real fields. Insertion of (4.1.21) into (4.1.19) gives

$$L_Y = -g\bar{\psi}_L \psi_R \sigma - g\bar{\psi}_R \psi_L \sigma - ig\bar{\psi}_L \tau^a \psi_R \pi_a + ig\bar{\psi}_R \tau^a \psi_L \pi_a \tag{4.1.22}$$

One has

$$\bar{\psi} P_L \psi = \psi^{\dagger} \gamma_0 P_L \psi_L = \psi^{\dagger} P_L \gamma_0 \psi = (P_R \psi)^{\dagger} \gamma_0 \psi_L \tag{4.1.23}$$

or

$$\bar{\psi} P_L \psi = \bar{\psi}_R \psi_L \tag{4.1.24}$$

and similarly

$$\bar{\psi} P_R \psi = \bar{\psi}_L \psi_R \tag{4.1.25}$$

Addition and subtraction of (4.1.24) and (4.1.25) give respectively [see (4.1.12)]

$$\bar{\psi}\psi = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi \tag{4.1.26}$$

and

$$\bar{\psi}\gamma_5\psi = \bar{\psi}_L\psi_R - \bar{\psi}_R\psi \qquad (4.1.27)$$

From (4.1.22), (4.1.26) and (4.1.27) follows for the Yukawa interaction Lagrangian

$$L_Y = -g\bar{\psi}\psi\sigma - ig\bar{\psi}\tau^a\gamma_5\psi\pi_a \qquad (4.1.28)$$

Finally we introduce a Lagrangian L_Σ for the spinless fields consisting of a kinetic energy term and a self-interaction term. This Lagrangian depends only on the spinless fields (4.1.21). The total Lagrangian L of the system is taken to be

$$L = L_0 + L_Y + L_\Sigma \qquad (4.1.29)$$

The Lagrangian L_Σ is taken to be invariant under $SU(2)_L \times SU(2)_R$. Since (4.1.18) gives

$$\Sigma^\dagger\Sigma \rightarrow \hat{\Sigma}^\dagger\hat{\Sigma} = U_R\Sigma^\dagger\Sigma\, U_R^\dagger \qquad (4.1.30)$$

it is clear that

$$Tr(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma), \quad Tr(\Sigma^\dagger\Sigma)^n \quad (n=0,1,2,...) \qquad (4.1.31)$$

are invariants. Furthermore

$$\partial_\mu\Sigma^\dagger\partial^\mu\Sigma = (\partial_\mu\sigma\partial^\mu\sigma + \sum_{a=1}^{3}\partial_\mu\pi_a\partial^\mu\pi_a)1 \qquad (4.1.32)$$

and

$$\Sigma^\dagger\Sigma = (\sigma^2 + \sum_{a=1}^{3}\pi_a^2)1 \qquad (4.1.33)$$

where 1 is the 2×2 unit matrix. In view of this we take for L_Σ

$$L_\Sigma = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\sum_{a=1}^{3}\partial_\mu\pi_a\partial^\mu\pi_a - V(\sigma^2 + \sum_{a=1}^{3}\pi_a^2) \qquad (4.1.34)$$

where the potential energy density is chosen to be

$$V(\sigma^2 + \pi^2) = \frac{\lambda}{4}[\sigma^2 + \pi^2 - F_\pi^2]^2 \qquad (4.1.35)$$

with

$$\pi^2 \equiv \sum_{a=1}^{3}\pi_a^2 \qquad (4.1.36)$$

Here λ is a dimensionless non-negative constant and F_π is a real constant with the dimension of a mass. The Hamiltonian density corresponding to (4.1.34) is obtained in the well-known way by a Legendre transformation and reads

$$H = \frac{1}{2}\{\pi_\sigma^2 + (\nabla\sigma)^2 + \sum_{a=1}^{3}(\pi_{\pi_a}^2 + (\nabla\pi_a)^2)\} + V(\sigma^2 + \pi^2) \qquad (4.1.37)$$

where the canonical conjugate momenta are defined by

$$\pi_\sigma = \frac{\partial L_\Sigma}{\partial \dot{\sigma}}, \quad \pi_{\pi_a} = \frac{\partial L_\Sigma}{\partial \dot{\pi}_a} \tag{4.1.38}$$

The coupling constant λ has to be non-negative in order that the energy is bounded from below. Whether the constant F_π^2 is zero or positive determines whether V has one or more minima. This has a large impact even on the qualitative behaviour of the system.

4.2 Spontaneous symmetry breaking and Goldstone bosons

The ground state of a system is by definition the field configuration with minimal energy. Hence it is the field configuration that has the minimal energy density in each spacetime point. From (4.1.37) it is then clear that the fields of the ground state are constants i.e. independent of space and time coordinates. Furthermore V has to be minimal. Hence a ground state satisfies

$$\sigma = \sigma_0, \quad \pi_a = \pi_{a0} \quad (\sigma_0, \pi_{a0} \text{ constant}) \tag{4.2.1}$$

and [see (4.1.35)]

$$\sigma_0^2 + \sum_a \pi_{a0}^2 = F_\pi^2 \tag{4.2.2}$$

From (4.2.1) one sees that the ground state is translation-invariant. According to the values taken by F_π^2 two distinct cases arise.

WIGNER-WEYL MODE. For $F_\pi = 0$ the potential energy density has precisely one ground state

$$\sigma_0 = 0, \quad \pi_{a0} = 0 \quad (a = 1,2,3) \tag{4.2.3}$$

The ground state is said to be non-degenerate. This case is called the Wigner-Weyl mode.

GOLDSTONE-NAMBU MODE. For $F_\pi \neq 0$ there is a plethora of ground states i.e. all constant fields (4.2.1) which satisfy (4.2.2). This situation is called the Goldstone-Nambu mode. Here the ground state is degenerate, that is their exist several different field configurations all with the same minimal energy. All these ground states are equivalent. They can be obtained from one another by means of a $SU(2)_L \times SU(2)_R$-transformation. If we choose a particular ground state for example

$$\sigma_0 = F_\pi, \quad \pi_{a0} = 0 \quad (a = 1.2.3) \tag{4.2.4}$$

then this ground state is not invariant under the whole group $SU(2)_L \times SU(2)_R$. Indeed the field Σ corresponding to (4.2.4) is given by $\Sigma_0 = \sigma_0 1$ (1 is the 2 × 2 unit matrix) and the latter field is invariant under (4.1.18) iff

$$U_L = U_R \tag{4.2.5}$$

Hence the ground state (4.2.4) is only invariant under a proper subgroup of

$SU(2)_L \times SU(2)_R$. This ground state of the system has less symmetry than the Lagrangian of the system and one says that there is *spontaneous symmetry breaking*. Equation (4.2.4) suggests to introduce the shifted fields

$$\sigma' = \sigma - F_\pi, \quad \pi'_a = \pi_a \tag{4.2.6}$$

which have the property

$$\sigma'_0 = 0, \quad \pi'_{a0} = 0 \tag{4.2.7}$$

Henceforth we suppress again the prime on the π-field. In terms of the shifted field the Lagrangian becomes

$$\begin{aligned} L = &\bar{\psi} i\gamma^\mu \partial_\mu \psi - gF_\pi \bar{\psi}\psi - g\sigma'\bar{\Psi}\psi + ig\bar{\psi}\tau^a\gamma_5\psi\pi_a \\ &+\frac{1}{2}(\partial_\mu\sigma'\partial^\mu\sigma' + \sum_a \partial_\mu\pi_a\partial^\mu\pi_a) - \frac{\lambda}{4}(\sigma'^2 + \sum_{a=1}^{3}\pi_a^2 + 2F_\pi\sigma')^2 \end{aligned} \tag{4.2.8}$$

Comparison of the first two terms in the right-hand side of (4.2.8) with the Dirac Lagrangian (4.1.1) tells us that we have here massive spin-1/2-particles with mass

$$m = gF_\pi \tag{4.2.9}$$

Although we started with a Dirac Lagrangian of massless spin-1/2-particles, these particles become massive due to the spontaneous symmetry breaking. This mechanism is called *mass generation.* Notice in this connection that the constant F_π determines the mass m and $F_\pi \neq 0$ gives rise to spontaneous symmetry breaking. Equation (4.2.9) is called the Goldberger-Treiman relation. The part of the Lagrangian which is quadratic in the fields σ' and π_a reads

$$L_q = \frac{1}{2}\partial_\mu\sigma'\partial^\mu\sigma' - \lambda F_\pi^2\sigma'^2 + \frac{1}{2}\sum_a \partial_\mu\pi_a\partial^\mu\pi_a \tag{4.2.10}$$

Hence the spinless field σ is massive with a mass given by

$$m_{\sigma'} = 2\lambda F_\pi^2 \tag{4.2.11}$$

and the spinless fields π_a are massless. The latter are called Goldstone bosons. It is again the spontaneous symmetry breaking which is the reason for the masslessness of the Goldstone bosons.

Summarizing, the Lagrangian (4.2.8) describes a system consisting of the protons and neutrons with a mass given by (4.2.9), massless neutral and charged pions π^0, π^+ and π^-, where

$$\pi^0 = \pi_3, \quad \pi^\pm = \frac{\pi_1 \pm i\pi_2}{\sqrt{2}} \tag{4.2.12}$$

and massive spinless particles, described by the field σ', with a mass given by (4.2.11). The nucleons interact with the pions via the fourth term appearing in the right-hand side of (4.2.8). The spinless particles interact with each other and have self-interactions via the last term in the right-hand side of (4.2.8).

Additional information about this chapter and the first section of the next section can be found in reference [3].

5 Nonlinear sigma models

In this chapter the sigma model of Gell-Mann and Lévy is extended by replacing the group $SU(2)$ by the group $SU(3)$. This leads us, more or less compulsary, to a so-called non-linear sigma model. The sigma model of Gell-Mannn and Lévy also has a non-linear version. Some of the symmetry of these models is removed by adding the Wess-Zumino term to its Lagrangian.

5.1 Chiral Lagrangians.

Point of departure in the sigma model of Gell-Mann and Lévy was the chiral symmetry $SU(2)_L \times SU(2)_R$ of the nucleon field (4.1.5). A more modern outfit of this model is obtained by replacing the proton and neutron field by the fields of the up-quark u and the down-quark d. Then (4.1.5) turns into

$$\psi = \begin{bmatrix} u \\ d \end{bmatrix} \tag{5.1.1}$$

Since there are more then two quark flavors this suggests the following generalization. By joining for instance the strange quark s (5.1.1) is replaced by

$$\psi = \begin{bmatrix} u \\ d \\ s \end{bmatrix} \tag{5.1.2}$$

The Lagrangian of these quarks, which are supposed again to be massless to begin with, reads [compare ((4.1.14) and (4.1.6)]

$$L_0 = \bar{p} i \gamma^\mu \partial_\mu p + \bar{n} i \gamma^\mu \partial_\mu n + \bar{s} i \gamma^\mu \partial_\mu s \tag{5.1.3}$$

or

$$L_0 = \bar{\psi} i \gamma^\mu \partial_\mu \psi \tag{5.1.4}$$

Introducing left-handed and right-handed components of the ψ-field [see (4.1.9)] the Lagrangian (5.1.4) gets, analogous to (4.1.15), the form

$$L_0 = \bar{\psi}_L i \gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R i \gamma^\mu \partial_\mu \psi_R \tag{5.1.5}$$

This Lagrangian is invariant under the transformations [compare (4.1.17)]

$$\psi_L \rightarrow \hat{\psi}_L = U_L \psi_L, \quad \psi_R \rightarrow \hat{\psi}_R = U_R \psi_R \tag{5.1.6}$$

where now U_L and U_R are arbitrary unitary 3×3-matrices. Hence this Lagrangian has a symmetry group $U(1)_L \times SU(3)_L \times U(1)_R \times SU(3)_R$. We concentrate on its subgroup $SU(3)_L \times SU(3)_R$.

The Σ-field of section 4.1 was a 2×2-matrix and this is now replaced by a 3×3-matrix which is also denoted by Σ. Its transformation rule under $SU(3)_L \times SU(3)_R$ reads [compare (4.1.18)]

$$\Sigma \rightarrow \hat{\Sigma} = U_L \Sigma U_R^\dagger \tag{5.1.7}$$

where $U_L, U_R \in SU(3)$. Now we deviate from the treatment of section 4.1.

There a reality condition was imposed on the fields appearing in Σ [see (4.1.21)]. The outcome was three Goldstone boson fields and one massive spinless field. Here we concentrate solely one the Goldstone bosons of the model suppressing the analogues of the σ-particles from the outset. This is effected by assuming that Σ is a unimodular unitary 3×3-matrix. Such a matrix requires eight real parameters for its parametrization. We choose as parametrization

$$\Sigma = \exp(2i \sum_{a=1}^{8} \pi_a \lambda_a / F_\pi) \tag{5.1.8}$$

where F_π is a real constant, $\lambda_a (a = 1,...,8)$ are the Gell-Mann matrices and $\pi_a (a = 1,...,8)$ are the eight real parameters of the $SU(3)$-matrix Σ. Since Σ is a field, i.e. Σ depends on x, the π_a's are also fields. They turn out to be the Goldstone fields of the model. The Gell-Mann matrices are rather conventional generators of the Lie algebra of $SU(3)$. They are hermitian traceless 3×3-matrices (see for instance chapter 17 of reference [4]). Notice that the matrices of the fields appearing in (5.1.7) are $SU(3)$ matrices. They can also be written in exponential form and then (5.1.7) reads

$$\Sigma = \exp(2i \sum_{a=1}^{8} \pi_a \lambda_a / F_\pi) \rightarrow \hat{\Sigma} = \exp(2i \sum_{a=1}^{8} \hat{\pi}_a \lambda_a / F_\pi) \tag{5.1.9}$$

where $\hat{\pi}_a (a = 1,...,8)$ are real fields since $\hat{\Sigma}$ is a $SU(3)$ matrix. The $SU(3)_L \times SU(3)_R$ transformations with $U_L = U_R$ are the ordinary $SU(3)$ transformations. They form a subgroup of $SU(3)_L \times SU(3)_R$. The $SU(3)_L \times SU(3)_R$ transformations with $U_L^\dagger = U_R$ are called pure chiral transformations. Under an ordinary $SU(3)$ transformation the transformation (5.1.7) gives rise to a linear transformation

$$\pi_a \rightarrow \hat{\pi}_a \tag{5.1.10}$$

For pure chiral transformations this is not the case, there one has

$$\pi_a \rightarrow \hat{\pi}_a = \alpha_a + \frac{1}{2} F_\pi c_a + \cdots \tag{5.1.11}$$

The Lagrangian of the Σ-field is constructed similar to the construction given in section 4.1. One takes

$$L_\Sigma = \frac{F_\pi^2}{16} Tr(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma) = \frac{1}{2} \sum_a \partial_\mu \pi_a \partial^\mu \pi_a + \cdots \tag{5.1.12}$$

Electromagnetic interaction can be introduced in this Lagrangian via minimal coupling i.e. the replacement of the partial derivatives ∂_μ by their gauge-covariant derivatives

$$D_\mu = \partial + ieQA_\mu \tag{5.1.13}$$

where A_μ is the electromagnetic potential and Q the charge matrix of the quarks

$$Q = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{-1}{3} & 0 \\ 0 & 0 & \frac{-1}{3} \end{bmatrix} \tag{5.1.14}$$

Notice that one has for the charges of the quarks $Q(u) = 2/3e$ and $Q(d) = Q(s) = -1/3e$. For the gauge-covariant derivative of the Σ-field one finds

$$D_\mu \Sigma = \partial_\mu \Sigma + ieA_\mu [Q, \Sigma] \tag{5.1.15}$$

Substitution of (5.1.15) into (5.1.12) gives a Lagrangian which can be used to describe electromagnetic interaction of spinless mesons. It was observed by Sutherland and Veltman in the context of the formal manipulations in current algebra that one cannot account for the electromagnetic decay $\pi^0 \to 2\gamma$ of the neutral pion. Likewise the Lagrangian

$$\tilde{L}_\Sigma = \frac{F_\pi^2}{16} Tr(D_\mu \Sigma^\dagger D^\mu \Sigma) = \frac{1}{2} \sum_a D_\mu \pi_a D^\mu \pi_a + \cdots \tag{5.1.16}$$

obtained from (5.1.12) by minimal coupling cannot account for the electromagnetic decay $\pi^0 \to 2\gamma$. The resolution of this problem lies in the fact that chiral symmetry associated with π_3 is broken by an anomaly. By an anomaly is meant the situation where the quantized theory has less symmetry than the corresponding classical theory. That such a thing can arise can be seen in the following ways. When the quantum field theory is described perturbatively by means of Feynman diagrams, the divergent diagrams need a regularization. Sometimes there does not exist a regularization which conserves all symmetries of the classical theory. In this way the quantized theory has fewer symmetries that the classical theory. At first sight it is perhaps somewhat puzzling how anomalies can arise in a quantization by means of path integrals. It is indeed the classical action which enters the path integral and this action has all the symmetries of the classical theory. Anomalies arise in that case when there does not exist a measure for the path integral which has all the symmetries of the classical theory.

In the present case the partial conservation law of the axial current is changed by the anomaly into

$$\partial_\mu j_{5\mu} = F_\pi m_\pi^2 \pi^0 - \frac{\alpha}{8\pi} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \tag{5.1.17}$$

It is the second term in the right-hand side of (5.1.17) which is due to the anomaly. It is not necessary to go through all kinds of quantum mechanical calculations to get this anomalous term. It can be obtained by adding a term to the Lagrangian. The additional term in the Lagrangian which gives rise to this anomaly is called the Wess-Zumino term. In the next section it is introduced in a very nice way following Witten (see reference [5]).

5.2 Wess-Zumino term à la Witten.

In this section we introduce the Wess-Zumino term. We follow here the elegant treatment by Witten (see reference [5]). The Lagrangian (5.1.12) is our starting point. All $\Sigma(x)$ are elements of the group $SU(3)$ and they are from now on denoted by

$$g(x) \equiv \Sigma(x) \tag{5.2.1}$$

The Lagrangian then reads

$$L_\Sigma = \frac{F_\pi^2}{16} Tr(\partial_\mu g^\dagger \partial^\mu g) \tag{5.2.2}$$

It is invariant under $SU(3)_L \times SU(3)_R$-transformations. The Lagrangian L_Σ is also invariant under spatial inversion $\mathbf{x} \rightarrow -\mathbf{x}$ giving rise to

$$\pi_a(t,\mathbf{x}) \rightarrow \hat{\pi}_a(t,\mathbf{x}) = -\pi_a(t,-\mathbf{x}) \tag{5.2.3}$$

or equivalently

$$g \rightarrow g^{-1}, \quad \mathbf{x} \rightarrow -\mathbf{x}, \quad t \rightarrow t \tag{5.2.4}$$

The Goldstone bosons are pseudoscalars and the partity transformations (5.2.3) and (5.2.4) are denoted by $\mathbf{P}$. The Lagrangian is also invariant under the naive parity transformation $\mathbf{P}_0$ given by:

$$g \rightarrow g, \quad \mathbf{x} \rightarrow -\mathbf{x}, \quad t \rightarrow t \tag{5.2.5}$$

Furthermore the Lagrangian is also invariant under the transformation

$$g \rightarrow g^{-1}, \quad \mathbf{x} \rightarrow \mathbf{x}, \quad t \rightarrow t \tag{5.2.6}$$

The latter transformation is equivalent to

$$\pi_a \rightarrow -\pi_a \tag{5.2.7}$$

and consequently it can be denoted by $(-1)^N$ were N is the number of bosons. Furthermore the Lagrangian (5.2.2) is invariant under the transformation

$$g \rightarrow g^T \tag{5.2.8}$$

For the pions in particular this gives

$$\pi^0 \rightarrow \pi^0, \quad \pi^+ \leftrightarrow \pi^-, \quad etc. \tag{5.2.9}$$

The transformation (5.2.9) interchanges particles and antiparticles and it is called particle-antiparticle conjugation or charge conjugation.

Quantum chromodynamics is like the above theory invariant under $\mathbf{P}$. Notice that $\mathbf{P} = \mathbf{P}_0 . (-1)^N$. However unlike the above theory quantum chromodynamics is not invariant under $\mathbf{P}_0$ and $(-1)^N$ separately. Our next goal is to violate both the latter symmetries by addition of a symmetry breaking term to the Lagrangian. The equation of motion corresponding to the Lagrangian (5.2.2) reads

$$\partial_\mu(g^{-1}\partial^\mu g) = 0 \tag{5.2.10}$$

A term which violates the symmetry $\mathbf{P}_0$ is readily incorporated in this equation:

$$\partial_\mu(g^{-1}\partial^\mu g) + \lambda\epsilon^{\mu\nu\kappa\lambda}g^{-1}(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}(\partial_\kappa g)g^{-1}(\partial_\lambda g) = 0 \quad (5.2.11)$$

The parity transformation **P** is still a symmetry of (5.2.11). We now turn to the question whether this equation can be derived from a Lagrangian. At first sight it seems hopeless to obtain the second term in the left-hand side of (5.2.11) from an interaction Lagrangian. The latter has to contain the Levi-Civita density $\epsilon^{\mu\nu\kappa\lambda}$ and the only pseudoscalar which comes to mind is equal to zero. Namely

$$\epsilon^{\mu\nu\kappa\lambda}Tr\{g^{-1}(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}(\partial_\kappa g)g^{-1}(\partial_\lambda g)\} = 0 \quad (5.2.12)$$

since the trace has cyclic symmetry. It is helpful to consider an analogous but simpler problem. Let us consider the motion of a particle of mass m constrained to a two-dimensional unit sphere. Introducing the constraint by means of a Lagrange multiplier λ the Lagrangian reads

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - \lambda(\mathbf{r}^2 - 1) \quad (5.2.13)$$

The Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\mathbf{r}}} - \frac{\partial L}{\partial\mathbf{r}} = 0, \quad \frac{d}{dt}\frac{\partial L}{\partial\dot{\lambda}} = \frac{\partial L}{\partial\lambda} = 0 \quad (5.2.14)$$

give rise to

$$m\ddot{\mathbf{r}} + 2\lambda\mathbf{r} = 0, \quad \mathbf{r}^2 = 1 \quad (5.2.15)$$

Elimination of λ gives the equation of motion

$$m\ddot{\mathbf{r}} + m\mathbf{r}\dot{\mathbf{r}}^2 = 0 \quad (5.2.16)$$

It is easily seen that this equation is invariant under time reversal **T** and spatial inversion **P**:

$$\mathbf{T}: \begin{array}{c} t \to -t \\ \mathbf{r} \to \mathbf{r} \end{array}, \quad \mathbf{P}: \begin{array}{c} t \to t \\ \mathbf{r} \to -\mathbf{r} \end{array} \quad (5.2.17)$$

We now modify the equation of motion (5.2.16) in such a way that it is no longer invariant under **P** and **T**, although it stays invariant under the combined transformation **PT**. The simplest modification satisfying these demands reads

$$m\ddot{\mathbf{r}} + m\mathbf{r}\dot{\mathbf{r}}^2 = \alpha\mathbf{r}\wedge\dot{\mathbf{r}} \quad (5.2.18)$$

At this point we ask the question, like in the case of the sigma model, whether this equation of motion can be derived from a Lagrangian or equivalently from an action. However in the present case the answer seems to be within reach. Firstly the right-hand side of this equation can be interpreted as the Lorentz force

$$\mathbf{F} = e[\mathbf{E}(\mathbf{r},t) + \frac{\dot{\mathbf{r}}}{c}\wedge\mathbf{B}(\mathbf{r},t)] \tag{5.2.19}$$

on an electric charge e with position $\mathbf{r} = \mathbf{r}(t)$ moving in the field of a magnetic monopole. The electric field strength $\mathbf{E}(\mathbf{x},t)$ and the magnetic induction $\mathbf{B}(\mathbf{x},t)$ of a magnetic monopole are given by

$$\mathbf{E}(\mathbf{x},t) = \mathbf{0}, \quad \mathbf{B}(\mathbf{x},t) = g\frac{\mathbf{x}}{4\pi|\mathbf{x}|^3} \tag{5.2.20}$$

where g is the magnetic charge of the monopole. Since the electric charge moves on the unit sphere one has $\mathbf{B}(\mathbf{r},t) = g\mathbf{r}/4\pi$ and this gives, using (5.2.19) and (5.2.20), indeed rise to the Lorentz force in the right-hand side of (5.2.18) by setting $\alpha = eg/4\pi$. Secondly the action of a (non-relativistic) particle in an electromagnetic field with potential ϕ and vector potential $\mathbf{A}$ reads

$$S = \int\{\frac{1}{2}m\dot{\mathbf{r}}^2 + e\phi(\mathbf{r},t) + e\dot{\mathbf{r}}.\mathbf{A}(\mathbf{r},t)\}\mathrm{d}t \tag{5.2.21}$$

So all we have to do is to determine the electromagnetic potentials of the magnetic monopole (5.2.20). The relations between the electromagnetic potentials and field strengths are given by

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\partial_t\mathbf{A}, \quad \mathbf{B} = \nabla\wedge\mathbf{A} \tag{5.2.22}$$

The potentials are taken to be time-independent since the field strengths are. Setting $\phi=0$ one gets $\mathbf{E}=\mathbf{0}$. There is however still the problem left of the determination of the vector potential $\mathbf{A}$ of the magnetic monopole (5.2.20). That is, one can try to find a solution of the equation

$$\nabla\wedge\mathbf{A} = g\frac{\mathbf{x}}{4\pi|\mathbf{x}|^3} \tag{5.2.23}$$

in the region outside the origin $\mathbf{x} = \mathbf{0}$ where it has a singularity. It is first shown that it has no solution. Then two alternatives towards a Lagrangian description are suggested. We now first show that the assumption that (5.2.23) has a solution leads to a contradiction by considering the magnetic flux. The magnetic flux Φ through a surface S (bounded by a curve $\gamma = \partial S$) is defined by

$$\Phi = \int_S \mathbf{B}.\mathrm{d}\mathbf{S} \tag{5.2.24}$$

Stokes' theorem gives [see second equation of (5.2.22)]

$$\Phi = \int_S \nabla\wedge\mathbf{A}.\mathrm{d}\mathbf{S} = \oint_\gamma \mathbf{A}.\mathrm{d}\mathbf{r} \tag{5.2.25}$$

where the well-known correspondence between the orientations of the surface integral and the contour integral are taken into account. Now we calculate the flux through the unit sphere S^2 around the magnetic monopole in two different ways. Let γ be a closed curve on this sphere which divides it into two parts D and D'. Then

$$\Phi(S^2) = \int_D \mathbf{B}.d\mathbf{S} + \int_{D'} \mathbf{B}.d\mathbf{S} = \oint_\gamma \mathbf{A}.d\mathbf{r} + \oint_{-\gamma} \mathbf{A}.d\mathbf{r} = 0 \tag{5.2.26}$$

since the contour integrals obtained from D and D' have opposite orientations. On the other hand (5.2.23) and (5.2.25) give

$$\Phi(S^2) = \frac{g}{4\pi} \int_{S^2} |\mathbf{x}|^{-3}\mathbf{x}.d\mathbf{S} = g \tag{5.2.27}$$

in contradiction with (5.2.26) for $g \neq 0$. Thus it seems that a magnetic monopole cannot be described by a vector potential. This jeopardizes the possibility of a Lagrangian description of the equation of motion (5.2.18). There are two ways out of this tangle. The first one is very elegant and is based on the insight that a vector potential of the magnetic monopole becomes a possibility if one abstains from its global representation as a vector field. One then turns to fibre bundle theory. The potential of a magnetic monopole can then be represented as the connection of a $U(1)$-bundle. In physical literature one most often takes recourse to the idea of the Dirac string. The contradiction caused by (5.2.27) is there evaded by changing the vector potential in such a way that the flux calculated in this formula also becomes zero. This change in the vector potential is however a clever one. This modified vector potential gives the magnetic induction of the monopole via (5.2.23) except on a curve which starts at the position of the monopole and runs to infinity. The vector potential is singular along this curve and the magnetic flux along this curve just compensates the magnetic flux outside this curve so that the total flux through a sphere around the monopole is equal to zero. Let us look for instance at the vector potential

$$\mathbf{A} = \frac{g}{4\pi} \frac{z}{r(r^2 - z^2)}(y, -x, 0) \tag{5.2.28}$$

where $r = |\mathbf{x}|$. This vector potential is singular in the origin $r = 0$ and along the positive z-axis $r = z$. It is easily verified that in the region where the potential (5.2.28) is regular its magnetic induction [see the second formula of (5.2.22)] is equal to that of the magnetic monopole (5.2.20). This vector potential can actually be interpreted as the vector potential of an infinitely thin solenoid with endpoints in the origin and infinity. The vector potential of an infinitely thin solenoid with endpoints in the origin and infinity but running along another curve describes outside this curve also the field of a magnetic monopole. Hence a Dirac monopole is a vector potential defined in the region outside a curve running from the magnetic monopole to infinity, such that the rotation of this vector potential gives the magnetic induction $\mathbf{B}$ given in (5.2.20). Hence the equation of motion (5.2.18) can be described by means of a Lagrangian if one excludes a curve which runs from the origin to infinity. This curve is called the Dirac string of the magnetic monopole and it can be chosen rather arbitrarily. For the motion of a classical particle with charge e in the field of a magnetic monopole one can choose the Dirac string to be disjoint with the trajectory of the charge thereby avoiding potential problems in the Lagrangian description. In quantum mechanics this does not work, since a wave function of a particle has to be defined everywhere. This troublesome feature of the Dirac string entails however a pleasant surprise. Let us consider a rather simple quantum mechanical quantity of our system: the partition function

$$Z = Tr\, \exp(-\beta H) \tag{5.2.29}$$

where H is the Hamiltonian of the system. The partition function Z can be expressed as a path integral

$$Z = \int_{\substack{\text{closed}\\ \text{paths}}} D\mathbf{r}(t) e^{-S_E[\mathbf{r},\beta]} \tag{5.2.30}$$

where S_E is the so-called Euclidean action

$$S_E[\mathbf{r},\beta] = \int_0^\beta \{\frac{1}{2} m\dot{\mathbf{r}}^2 - ie\dot{\mathbf{r}}.\mathbf{A}(\mathbf{r},t)\}dt \tag{5.2.31}$$

and the path integral runs through the set of closed paths with $\mathbf{r}(0) = \mathbf{r}(\beta)$. Notice that the Euclidean action is obtained from the action S by a Wick rotation $t \rightarrow -it$. Insertion of (5.2.31) into (5.2.30) gives the following problematic factor in the integrand of the path integral

$$I = \exp(ie\int_0^\beta \mathbf{A}.\dot{\mathbf{r}}dt) = \exp(ie\oint_\gamma \mathbf{A}.d\mathbf{r}) \tag{5.2.32}$$

where γ is a closed curve on the unit sphere. The problem is that the contour integral is ill-defined because of the Dirac string. Although application of Stokes' law gives

$$I = \exp(ie\int \mathbf{B}.d\mathbf{S}) \tag{5.2.33}$$

I is still ill-defined since there is no preferred choice for the surface bounded by the curve γ. The disks D and D' into which the curve γ divides the unit sphere are equally good. In order to eliminate this arbitrariness we require that both choices give the same result, i.e.

$$\exp(ie\int_D \mathbf{B}.d\mathbf{S}) = \exp(-ie\int_{D'} \mathbf{B}.d\mathbf{S}) \tag{5.2.34}$$

Since $D \cup D' = S^2$ this gives

$$\exp(ie\int_{S^2} \mathbf{B}.d\mathbf{S}) = 1 \tag{5.2.35}$$

or [see (5.2.27)]

$$\exp(i.e.g) = 1 \tag{5.2.36}$$

From this follows Dirac's quantization condition for the electric charge. It reads

$$e.g = 2\pi n \quad (n = 0, \pm 1, \pm 2, ...) \tag{5.2.36}$$

Notice that we have used natural units where $h/2\pi = 1$, otherwise the right-hand side would have to be multiplied by $h/2\pi$. From (5.2.36) follows that the coupling constant $\alpha = eg/2\pi$ in (5.2.18) is quantized (discrete):

$$\alpha = n/2 \quad (n = 0, \pm 1, \pm 2, \cdots) \tag{5.2.37}$$

Inspired by this example we now determine a term which breaks the $\mathbf{P}_0$ invariance of (5.2.2) when added to this Lagrangian. Spacetime in however first transformed into a four-dimensional euclidean space by means of a Wick rotation. Thereafter the latter space is compactified by replacing it by a four-dimensional sphere S^4. The field g [see (5.2.1)] is then a map

$$g: S^4 \rightarrow SU(3) \tag{5.2.38}$$

Notice that

$$\pi_4(SU(3)) = 0 \tag{5.2.39}$$

and $g(S^4)$ is the boundary of a five-dimensional disk. The integral in the left-hand side of (5.2.34) can be written as

$$\int_D \mathbf{B}.d\mathbf{S} = \int_D F_{ij} d\sigma^{ij} \tag{5.2.40}$$

Its analogue in the present case is

$$\Gamma = \int_Q \omega_{ijklm} d\sigma^{ijklm} \tag{5.2.41}$$

Similar to (5.2.34) and (5.2.35) one is led to

$$\int_{Q \cup Q'} \omega_{ijklm} d\sigma^{ijklm} = 2\pi n \quad (n = 0, \pm 1, \cdots) \tag{5.2.42}$$

where

$$\omega_{ijklm} d\sigma^{ijklm} = \tag{5.2.43}$$

$$-\frac{i}{240\pi^2} Tr[g^{-1}\frac{\partial g}{\partial y^i} g^{-1}\frac{\partial g}{\partial y^j} g^{-1}\frac{\partial g}{\partial y^k} g^{-1}\frac{\partial g}{\partial y^l} g^{-1}\frac{\partial g}{\partial y^m}] d\sigma^{ijklm}$$

with (y^i) coordinates on Q. By means of Stokes' theorem the symmetry-breaking part of the action Γ [see (5.2.41)] can be written as an integral over spacetime $\partial Q = S^4$. Finally the Wess-Zumino action obtained in this way reads

$$S = \frac{F_\pi^2}{16} \int d^4x Tr(\partial_\mu g^\dagger \partial^\mu g) + n\Gamma \tag{5.2.44}$$

where $n \in \mathbb{Z}$.

6 BOSONIZATION

In the early sixties Skyrme (see reference [6] and the other references cited there) tried to construct a theory of a self-interacting boson field, describing besides mesons also nucleons. The latter are fermions. Essentially all this amounts to an attempt to construct a fermion field theory from a boson field theory. A satisfactory realization of this goal is possible in two-dimensional spacetime. A well-known example is the equivalence of the quantum mechanical sine-Gordon theory and the massive Thirring model (see references [7] and [8]). Coleman establishes the equivalence between these two theories by comparison of their Green's functions. Mandelstam on the other hand expresses the fermion field in terms of the boson field. The latter approach becomes quite transparent when the energy-momentum tensor of the fermion field theory is represented in the so-called Sugawara form. The latter is another achievement also obtained in the early sixties. One of the efforts to construct a dynamical theory of elementary particles was then based on so-called current algebras. At that time currents were thought to be nearer to physics then quantum fields. This was due to the failure of formulating a quantum field theory of weak and strong interactions. However when currents become the primary objects it may be suspected that the energy-momentum tensor can also be expressed in terms of currents. This was achieved by Sugawara and Sommerfield. In the next two sections this construction will be exposed for a massless Dirac field in two-dimensional spacetime (see also references [9], [10] and [11]).

6.1 Sugawara energy-momentum tensor

The Lagrangian of a massless Dirac field ψ in two-dimensional spacetime reads

$$L = \frac{i}{2}[\bar{\psi}\gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi})\gamma^\mu \psi] \qquad (6.1.1)$$

where the Dirac matrices γ^μ are 2×2-matrices characterized by the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad (\mu, \nu = 0, 1) \qquad (6.1.2)$$

and the Minkowski metric is taken to be

$$\eta^{00} = -\eta^{11} = 1, \quad \eta^{01} = \eta^{10} = 0. \qquad (6.1.3)$$

More explicitly we can choose

$$\gamma^0 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \gamma^1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \qquad (6.1.4)$$

and define

$$\gamma_5 = \gamma^0\gamma^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \qquad (6.1.5)$$

The transformation

$$\psi \to \hat{\psi} = e^{i\epsilon}\psi, \quad \bar{\psi} \to \hat{\bar{\psi}} = e^{-i\epsilon}\bar{\psi} \tag{6.1.6}$$

with $\epsilon \in \mathbb{R}$ is a symmetry transformation. Hence the corresponding Noether current is conserved. The corresponding current operator is obtained by replacing the Dirac fields by Dirac operator fields (quantization) and normal ordening the resulting expression. This gives for the current operators

$$j^\mu(t,x) = :\bar{\psi}\gamma^\mu\psi:. \tag{6.1.7}$$

Their equal-time commutation relations read

$$[j^\mu(t,x), j^\nu(t,y)] = \frac{-i}{\pi}\epsilon^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y) \tag{6.1.8}$$

where

$$\epsilon^{01} = -\epsilon^{10} = 1, \quad \epsilon^{00} = \epsilon^{11} = 0. \tag{6.1.9}$$

Translations in spacetime are also symmetry transformations. The corresponding conserved Noether current is the energy-momentum tensor. The energy-momentum tensor operator is again obtained by quantization and normal ordering. It reads

$$\Theta^{\mu\nu} = \frac{i}{2}:[\bar{\psi}\gamma^\mu\partial^\nu\psi - (\partial^\nu\bar{\psi})\gamma^\mu\psi]:. \tag{6.1.10}$$

Following Belinfante and Rosenfeld this energy-momentum tensor can be symmetrized. This symmetric energy-momentum tensor of Belinfante and Rosenfeld is the energy-momentum tensor which in general relativity is the source of the gravitational field. It reads

$$\begin{aligned} T^{\mu\nu} =: \frac{i}{4}(\bar{\psi}\gamma^\mu\partial^\nu\psi + \bar{\psi}\gamma^\nu\partial^\mu\psi - (\partial^\nu\bar{\psi})\gamma^\mu\psi \\ -(\partial^\mu\bar{\psi})\gamma^\nu\psi - \eta^{\mu\nu}\bar{\psi}i\gamma^\rho\partial_\rho\psi:. \end{aligned} \tag{6.1.11}$$

By means of Wick's theorem it can be shown that this energy-momentum tensor can be expressed in terms of the current operator (see section IVB of reference [10]). This so-called Sugawara form of the energy-momentum tensor is given by

$$T^{\mu\nu} = \frac{\pi}{2}(j^\mu j^\nu + j^\nu j^\mu - \eta^{\mu\nu}j_\rho j^\rho) \tag{6.1.12}$$

where, suppressing for a moment their time-dependence, the products of current operators are defined by

$$\begin{aligned} j^\mu(x)j^\nu(x) = \\ \lim_{\epsilon\to 0}\{j^\mu(x+\tfrac{1}{2}\epsilon)j^\nu(x-\tfrac{1}{2}\epsilon) - <0|j^\mu(x+\tfrac{1}{2}\epsilon)j^\nu(x-\tfrac{1}{2}\epsilon)|0>\}. \end{aligned} \tag{6.1.13}$$

The energy-momentum vector is defined in the usual way by

$$P^\mu = \int T^{0\mu}(t,x)\mathrm{d}x. \tag{6.1.14}$$

By means of (6.1.13), (6.1.14) and the Heisenberg equation

$$i\partial_\mu A = [A, P_\mu] \tag{6.1.15}$$

one gets

$$\partial_0\psi = -i\pi[j^0 + \gamma_5 j^1]\psi \tag{6.1.16}$$

and

$$\partial_1\psi = i\pi[j^1 + \gamma_5 j^0]\psi \tag{6.1.17}$$

From (6.1.16) and (6.1.17) follows of course the Dirac equation corresponding to the Lagrangian (6.1.1). It reads

$$i\gamma^\mu\partial_\mu\psi = 0. \tag{6.1.18}$$

6.2 Boson formulation of two-dimensional Dirac theory

Formal integration of (6.1.17) gives

$$\psi(t,x) = \exp\{i\pi\int[j^1(t,x') + \gamma_5 j^0(t,x')]dx'\}\psi_0 \tag{6.2.1}$$

where ψ_0 is a constant spinor. Next we take a closer look at the currents in the right-hand side of (6.2.1). The vector current (6.1.7) of a free Dirac field is conserved, i.e.

$$\partial_0 j^0 + \partial_1 j^1 = 0. \tag{6.2.2}$$

Hence there exists a scalar field ϕ such that

$$j^0 = \frac{1}{\sqrt{\pi}}\partial_1\phi, \quad j^1 = -\frac{1}{\sqrt{\pi}}\partial_0\phi. \tag{6.2.3}$$

The choice of the normalization factors in (6.2.3) is such that the scalar field ϕ satisfies the canonical commutation relations. Indeed, inserting (6.2.3) in (6.1.8) gives

$$[\partial_1\phi(x), \pi(y)] = i\frac{\partial}{\partial x}\delta(x-y) \quad (\pi \equiv \partial_0\phi) \tag{6.2.4}$$

which is compatible with the canonical commutation relation

$$[\phi(x), \pi(y)] = i\delta(x-y). \tag{6.2.5}$$

Insertion of (6.2.3) into (6.2.1) gives

$$\begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix} = \begin{pmatrix} C_+ \exp i\sqrt{4\pi}\,\phi_+ \\ C_- \exp i\sqrt{4\pi}\,\phi_- \end{pmatrix} \tag{6.2.6}$$

where C_+ and C_- are constants and

$$\phi_\pm(t,x) = \frac{1}{2}\left[\phi(t,x) \mp \int_{-\infty}^{x}\partial_0\phi(t,x')dx'\right]. \tag{6.2.7}$$

Like (6.2.1) also (6.2.6) is still ill-defined. This is remedied by the introduction of an ultraviolet cutoff Λ. Let the creation and annihilation operators be introduced as usual by

$$\phi(t,x) = \int_{-\infty}^{\infty} \frac{dk}{4\pi k^0}[a_k e^{-ik_\mu x^\mu} + a_k^\dagger e^{ik_\mu x^\mu}] \tag{6.2.8}$$

then the chiral fields are defined by

$$\phi_\pm(t,x) = \int_0^{\pm\infty} \frac{dk}{4\pi k^0}[a_k e^{-ik_\mu x^\mu} + a_k^\dagger e^{ik_\mu x^\mu}]e^{-|k|/2\Lambda}. \tag{6.2.9}$$

At the end of all calculations the limit $\Lambda \to \infty$ is taken. The commutation relations of the chiral fields read

$$[\phi_+(x),\phi_+(y)] = \frac{i}{4}\epsilon(x-y) \tag{6.2.10}$$

$$[\phi_-(x),\phi_-(y)] = -\frac{i}{4}\epsilon(x-y) \tag{6.2.11}$$

$$[\phi_+(x),\phi(y)] = \frac{i}{4} \tag{6.2.12}$$

where

$$\epsilon(x) = \begin{cases} 1 & (x>0) \\ 0 & (x=0) \\ -1 & (x<0) \end{cases} \tag{6.2.13}$$

Finally (6.2.6) can be written as

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \left(\frac{\Lambda}{2\pi}\right)^{\frac{1}{2}} \begin{pmatrix} \exp i\sqrt{4\pi}\,\phi_+ \\ \exp i\sqrt{4\pi}\,\phi_- \end{pmatrix} \tag{6.2.14}$$

Next it is shown that ϕ is a free massless Klein-Gordon field and thus (6.2.14) expresses the Dirac field in terms of a Bose field [see (6.2.5)]. For a massless Dirac field the axial vector current defined by

$$j_A^\mu =: \bar{\psi}\gamma^\mu\gamma_5\psi: \tag{6.2.15}$$

is conserved as well as the vector current (6.1.7). That is

$$\partial_\mu j_A^\mu = 0. \tag{6.2.16}$$

The identity

$$\gamma^\mu\gamma_5 = \gamma_\nu\epsilon^{\nu\mu} \tag{6.2.17}$$

together with (6.1.7) and (6.2.15) gives

$$j_A^\mu = j_\nu\epsilon^{\nu\mu}. \tag{6.2.18}$$

Inserting (6.2.18) in (6.2.16) gives

$$\partial_0 j^1 + \partial_1 j^0 = 0. \tag{6.2.19}$$

From (6.2.3) and (6.2.19) finally follows that ϕ satisfies the free massless Klein-Gordon equation

$$(\partial_0^2 - \partial_1^2)\phi = 0. \tag{6.2.20}$$

The anticommutativity of the Dirac field can be derived from the operator identity

$$e^A e^B = e^B e^A e^{[A,B]} \tag{6.2.21}$$

which holds whenever

$$[A,[A,B]] = [[A,B],B] = 0. \tag{6.2.22}$$

This gives

$$\begin{aligned} \psi_+(x)\psi_+(y) &= \frac{\Lambda}{2\pi} e^{i\sqrt{4\pi}\phi_+(x)} e^{i\sqrt{4\pi}\phi_+(y)} \\ &= -\frac{\Lambda}{2\pi} e^{i\sqrt{4\pi}\phi_+(y)} e^{i\sqrt{4\pi}\phi_+(x)} \\ &= -\psi_+(y)\psi_+(x). \end{aligned} \tag{6.2.23}$$

A similar relation holds for the suffix + replaced by −. Hence

$$\{\psi_a(x),\psi_b(y)\} = 0 \quad (a,b=\pm). \tag{6.2.24}$$

The derivation of the remaining anticommutation relation requires a consideration of the short-distance behavior. One finds

$$\{\psi_+(x),\psi_+^\dagger(y)\} = \frac{1}{\pi} \lim_{\Lambda\to\infty} \frac{\Lambda^{-1}}{\Lambda^{-2}+(x-y)^2} \tag{6.2.25}$$

or slightly more general

$$\{\psi_a(x),\psi_b^\dagger(y)\} = \delta(x-y)\delta_{ab} \quad (a,b=\pm). \tag{6.2.26}$$

In the next section bosonization will be generalized to a system of free fermions with a non-Abelian symmetry group. For this it is useful to express the current (6.2.3), i.e.

$$j^\mu = \frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \phi, \tag{6.2.27}$$

in terms of elements of the unitary group $U(1)$ defined by

$$g = \exp(i\sqrt{4\pi}\,\phi). \tag{6.2.28}$$

This gives

$$j^\mu = -\frac{i}{2\pi} \epsilon^{\mu\nu} g^{-1} \partial_\nu g \tag{6.2.29}$$

or

$$j^{\mu} = -\frac{i}{2\pi}\epsilon^{\mu\nu}(\partial_{\nu}g)g^{-1}. \tag{6.2.30}$$

Both alternatives are equally good in this case. However in the next section g will be an element of a non-abelian group and then this is no longer the case.

6.3 Non-Abelian bosonization

In this section the bosonization of the free fermion theory with N Majorana fields ψ^k $(k=1,...,N)$ is discussed. The action of this theory reads

$$S = \int \frac{1}{2}\bar{\psi}_k i\gamma^{\mu}\partial_{\mu}\psi^k d^2x \tag{6.3.1}$$

where $\bar{\psi} \equiv \psi^{\dagger}\gamma^0$. A convenient representation of the Dirac matrices is

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6.3.2}$$

and this gives

$$\gamma_5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.3.3}$$

In this representation a Majorana field is characterized by being real-valued. The action (6.3.1) can be separated into parts for the left-moving and the right-moving fermions respectively since

$$\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi = \psi^T(\partial_0 + \gamma_5\partial_1)\psi \tag{6.3.4}$$

where ψ is the column matrix (ψ^k). Indeed defining

$$\psi_+ = \frac{1}{2}(1-\gamma_5)\psi, \; \psi_- = \frac{1}{2}(1+\gamma_5)\psi \tag{6.3.5}$$

gives

$$\gamma_5\psi_+ = -\psi_+, \; \gamma_5\psi_- = \psi_- \tag{6.3.6}$$

Furthermore (6.3.4) gives

$$\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi = \psi_+(\partial_0 - \partial_1)\psi_+ + \psi_-(\partial_0 + \partial_1)\psi_-. \tag{6.3.7}$$

Hence

$$S = \frac{1}{2}i\int\{\psi_+^k(\frac{\partial}{\partial t} - \frac{\partial}{\partial x})\psi_+^k + \psi_-^k(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})\psi_-^k\}d^2x. \tag{6.3.8}$$

Its equations of motion read

$$(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x})\psi_{\mp} = 0. \tag{6.3.9}$$

Introducing light-cone coordinates

$$x^{\pm} = (x^0 \pm x^1)/\sqrt{2} \tag{6.3.10}$$

gives

$$\frac{\partial\psi_-}{\partial x^+} = 0 = \frac{\partial\psi_+}{\partial x^-} \tag{6.3.11}$$

and one sees that ψ_+ only depends on x^+ and ψ_- is only a function of x_-. Thus ψ_+ describes left-moving modes and ψ_- right-moving modes. The Lagrangian L appearing in (6.3.8) is invariant under the transformations

$$\psi_- \to A\psi_-, \quad \psi_+ \to B\psi_+ \tag{6.3.12}$$

where A and B are orthogonal $N\times N$-matrices. Hence the symmetry group of the Lagrangian is $O(N)\times O(N)$. This group is also denoted by $O_L(N)\times O_R(N)$, where L and R signal the left- and right-handedness of the corresponding fields. This invariance group gives rise to the following conserved currents of the Lagrangian (6.3.8)

$$J_-^{kl} = -\psi_-^k\psi_-^l, \; J_+^{kl} = -\psi_+^k\Psi_+^l. \tag{6.3.13}$$

The conservation laws read [compare (6.3.11)]

$$\partial_+ J_-^{kl} = 0, \;\; \partial_- J_+^{kl} = 0. \tag{6.3.14}$$

Under the transformation (6.3.12) the currents transform as

$$J_-^{kl} \to A_m^k A_n^l J_-^{mn} = (AJ_-A^T)^{kl} \tag{6.3.15}$$

and

$$J_+^{kl} \to B_m^k B_n^l J_+^{mn} = (BJ_+B^T)^{kl}. \tag{6.3.16}$$

Although below the indices of these currents will often be suppressed without further notice it is sometimes convenient to collect them in one expression. Let $M=(M_{kl})$ be an arbitrary $N\times N$ matrix then we can collect the components of the currents in the quantity $Tr(MJ_\pm)$. The commutation relations of the currents then can be expressed by [compare (6.1.8)]

$$[Tr\{MJ_+\},Tr\{NJ_+\}] = 2i\hbar\delta(\sigma-\sigma')Tr\{[M,N]J_+\} - \frac{n}{\pi}i\hbar\delta'(\sigma-\sigma')Tr(MN) \tag{6.3.17}$$

$$[Tr\{MJ_-\},Tr\{NJ_-\}] = 2i\hbar\delta(\sigma-\sigma')Tr\{[M,N]J_-\} + \frac{n}{\pi}i\hbar\delta'(\sigma-\sigma')Tr(MN) \tag{6.3.18}$$

and

$$[Tr\{MJ_+\}, \; Tr\{NJ_-\}] = 0 \tag{6.3.19}$$

Let us now slightly generalize all this. Instead of fermion fields ψ^k $(k=1,...,N)$ one can take fermion fields with an extra label a. That is, we take the fermion fields ψ^{ka} $(a=1,..., \mathrm{k})$. One defines

$$J_\pm^{kl} = -\sum_{a=1}^{\mathrm{k}} \psi_\pm^{ka}\psi_\pm^{la} \tag{6.3.20}$$

The commutation relation (6.3.18) then becomes

$$[Tr\{MJ_-\},\ Tr\{NJ_-\}] = 2i\hbar\delta(\sigma-\sigma')Tr\{[M,N]J_-\} + \frac{n}{\pi}\mathrm{k}i\hbar\delta'(\sigma-\sigma')Tr(MN) \tag{6.3.21}$$

where now the constant k appears in the Schwinger term in the right-hand side of (6.3.21).

We now wish to express the currents (6.3.13) in terms of bose fields. Inspired by (6.2.29) and (6.2.30) one takes a bose field g which takes its values in the symmetry group $O_L(N)\times O_R(N)$ of the Lagrangian. Which of these two, now inequivalent, alternatives (6.2.29) and (6.2.30) should one follow? The choice

$$J_+ \propto g^{-1}\partial_+ g,\quad J_- \propto g^{-1}\partial_- g \tag{6.3.22}$$

is inconsistent for a non-abelian group. Indeed insertion of (6.3.22) in (6.3.14) gives

$$g^{-1}\partial_+\partial_- g + \partial_+ g^{-1}\partial_- g = 0 \tag{6.3.23}$$

and

$$g^{-1}\partial_+\partial_- g + \partial_- g^{-1}\partial_+ g = 0 \tag{6.3.24}$$

or

$$(\partial_+ g^{-1})\partial_- g = (\partial_- g^{-1})\partial_+ g. \tag{6.3.25}$$

The latter equation is easily seen to be inconsistent by insertion of

$$g = \exp i\,(x^- T_1 + x^+ T_2) \tag{6.3.26}$$

where T_1 and T_2 are generators of the non-abelian group, and setting $x^- = x^+ = 0$. For the choice

$$J_+ = \frac{i}{2\pi}g^{-1}\partial_+ g,\quad J_- = -\frac{i}{2\pi}(\partial_- g)g^{-1} \tag{6.3.27}$$

this inconsistency does not appear since their respective conservation laws

$$0 = \partial_-(g^{-1}\partial_+ g) = g^{-1}[\partial_-\partial_+ g + g(\partial_- g^{-1})\partial_+ g] \tag{6.3.28}$$

and

$$0 = \partial_+[(\partial_- g)g^{-1}] = [\partial_-\partial_+ g + (\partial_- g)(\partial_+ g^{-1})g]g^{-1} \tag{6.3.29}$$

are now shown to be equivalent. Indeed one has the identity

$$0 = \partial_\pm(gg^{-1}) = \partial_\pm g)g^{-1} + g(\partial_\pm g^{-1}) \tag{6.3.30}$$

or

$$\partial_\pm g^{-1} = -g^{-1}(\partial_\pm g)g^{-1}. \tag{6.3.31}$$

This implies

$$g(\partial_- g^{-1})\partial_+ g = -(\partial_- g)g^{-1}\partial_+ g = (\partial_- g)(\partial_+ g^{-1})g. \tag{6.3.32}$$

and this shows that the conservation laws (6.3.28) and (6.3.29) are indeed equivalent.

We now turn to the following question. What is the Lagrangian of the field g such that the resulting bose field theory is equivalent to the fermi field theory with the Lagrangian (6.3.1)? Obviously this Lagrangian must also have the symmetry group $O_L(N)\times O_R(N)$. To implement this the field g is assumed to transform according to a representation of $O_L(N)\times O_R(N)$ and one constructs an invariant Lagrangian from this field. The transformation rule of the field g under $(A,B)\in O_L(N)\times O_R(N)$ is taken to be

$$g \rightarrow AgB^{-1}. \tag{6.3.33}$$

Insertion of (6.3.33) into (6.3.27) indeed gives [compare (6.2.14) and (6.2.15)]

$$J_- \rightarrow AJ_-A^{-1} \tag{6.3.34}$$

and

$$J_+ \rightarrow BJ_+B^{-1}. \tag{6.3.35}$$

The transformation rule (6.3.33) is similar to (5.1.7) and in analogy with (5.2.2) [see also (5.2.1)] one is tempted to take as action

$$S = \frac{1}{4\lambda^2}\int d^2x Tr(\partial_\mu g \partial^\mu g). \tag{6.3.36}$$

This action is manifestly invariant under the transformations (6.3.33). However this theory cannot be equivalent to the free fermion theory (6.3.1) for a number of reasons (see reference [12]). For instance the Lagrangian L of the action (6.3.31) is invariant under the naive parity transformation $\mathbf{P}_0$ [see (5.2.5)] whereas the Lagrangian (6.3.8) of the free fermion theory is certainly not invariant under the naive parity transformation

$$\psi_\pm \rightarrow \psi_\pm, \quad x \rightarrow -x, \quad t \rightarrow t. \tag{6.3.37}$$

Recall that the Lagrangian of (6.3.36) is invariant under the parity transformation $\mathbf{P}$ [see (5.2.6) and (5.2.7)] whereas the Lagrangian of the free fermion theory (6.3.8) is invariant under the parity transformation

$$\psi_\pm \rightarrow \psi_\mp, \quad x \rightarrow -x, \quad t \rightarrow t. \tag{6.3.38}$$

Notice that this entails

$$\psi(x,t) = \begin{bmatrix}\psi_- \\ \psi_+\end{bmatrix} \rightarrow \begin{bmatrix}\psi_+(-x,t) \\ \psi_-(-x,t)\end{bmatrix} = \gamma^0\psi(-x,t). \tag{6.3.39}$$

All this is similar to the situation we encountered in section 5.2. So we introduce here also a Wess-Zumino term [compare (5.2.41) and (5.2.43)]. Spacetime is here two-dimensional and it is taken to be a two-dimensional sphere S^2 in the Euclidean treatment. The field g is then a map [compare (5.2.38)]

$$g : S^2 \rightarrow O(N). \tag{6.3.40}$$

Notice that [compare (5.2.39)]

$$\pi_2(O(N)) = 0 \tag{6.3.41}$$

and consequently the mapping (6.3.40) can be extended to a mapping of the solid sphere (ball) **B**, consisting of S^2 and its interior, into $O(N)$:

$$\bar{g} : \mathbf{B} \to O(N). \tag{6.3.42}$$

Let (y^1,y^2,y^3) be a coordinate system overlapping **B**. Then the Wess-Zumino term is given by [compare again (5.2.41) and (5.2.43)]

$$\Gamma = \frac{1}{24\pi}\int d^3y\epsilon^{ijk}\, Tr(\bar{g}^{-1}\frac{\partial \bar{g}}{\partial y^i}\bar{g}^{-1}\frac{\partial \bar{g}}{\partial y^j}\bar{g}^{-1}\frac{\partial \bar{g}}{\partial y^k}). \tag{6.3.43}$$

Actually (6.3.43) does not define Γ unambiguously for a given field g since the latter can be extended to $\bar{g}$ in topologically inequivalent ways. These topologically inequivalent extension are classified by

$$\pi_3(O(N)) \approx \mathbf{Z} \tag{6.3.44}$$

and Γ is only defined modulo 2π. Analogous to (5.2.44) we get from (6.3.36) and (6.3.43) the action

$$S = \frac{1}{4\lambda^2}\int d^2x\; Tr(\partial_\mu g \partial^\mu g^{-1}) + n\Gamma. \tag{6.3.45}$$

Since Γ is only defined modulo 2π quantization of the theory requires n to be an integer [compare (5.2.36)]:

$$n \in \mathbf{Z}. \tag{6.3.46}$$

The bose field theory with an action (6.3.45) which satisfies (6.3.46) is called the Wess-Zumino-Witten model (WZW-model). We will investigate in the next section under what conditions it is equivalent to fermi field theory (6.3.1).

6.4 Kac-Moody algebra of the W.Z.W.-model

We first calculate the equations of motion of the Wess-Zumino-Witten model. This is done by means of Hamilton's action principle. The action (6.3.45) is a functional of the $O(N)$-valued field g. Let $g(\lambda)(\lambda\in\mathbb{R})$be a one-parameter family of such fields with

$$g(\lambda) = g + \frac{\partial g(\lambda)}{\partial\lambda}|_{\lambda=0}\lambda + O(\lambda^2) \tag{6.4.1}$$

Then Hamilton's action principle asserts that the equations of motion follow from requiring

$$\frac{\partial}{\partial\lambda}S[g(\lambda)]|_{\lambda=0} = 0 \tag{6.4.2}$$

for all

$$g' := \frac{\partial g(\lambda)}{\partial\lambda}|_{\lambda=0} \tag{6.4.3}$$

Since [see (6.3.45)]

$$\frac{\partial}{\partial\lambda}S[g(\lambda)]|_{\lambda=0}=\frac{1}{2\lambda^2}\int d^2x\, Tr\{g^{-1}g'\partial_\mu(g^{-1}\partial^\mu g)\} \qquad (6.4.4)$$
$$-\frac{n}{8\pi}\int d^2x\, Tr\{g^{-1}g'\epsilon^{\mu\nu}\partial_\mu(g^{-1}\partial_\nu g)\}$$

the equation of motion reads

$$\frac{1}{2\lambda^2}\partial_\mu(g^{-1}\partial^\mu g)-\frac{n}{8\pi}\epsilon^{\mu\nu}\partial_\mu(g^{-1}\partial_\nu g)=0 \qquad (6.4.5)$$

or

$$\left[\frac{1}{2\lambda^2}+\frac{n}{8\pi}\right]\partial_-(g^{-1}\partial_+g)+\left[\frac{1}{2\lambda^2}-\frac{n}{8\pi}\right]\partial_+(g^{-1}\partial_-g)=0 \qquad (6.4.6)$$

where

$$\sigma\equiv x^-:=\frac{x^0-x^1}{\sqrt{2}},\quad \tau\equiv x^+:=\frac{x^0+x^1}{\sqrt{2}} \qquad (6.4.7)$$

are the so-called light cone coordinates. From (6.4.6) one sees that one gets the desired equation (6.3.28), that is

$$\partial_-(g^{-1}\partial_+g)=0 \qquad (6.4.8)$$

for the choice

$$\lambda^2=\frac{4\pi}{n} \qquad (6.4.9)$$

The action of the W.Z.W.-model becomes for this choice

$$S=\frac{n}{16\pi}\int d^2x\; Tr(\partial_\mu g\partial^\mu g^{-1})+n\Gamma \qquad (6.4.10)$$

We now show that the general solution of (6.4.8) is given by

$$g(x^+,x^-)=A(x^-)B(x^+) \qquad (6.4.11)$$

where A and B are $O(N)$-valued functions. The expression between the parentheses in the left-hand side of (6.4.8) is a function $F=F(x^+)$. Hence

$$\partial_+g=g\,F(x^+) \qquad (6.4.12)$$

From $g\in O(N)$ follows that F is anti-symmetric. Indeed

$$\mathbb{O}=\partial_+(g^Tg)=(\partial_+g^T)g+g^T\partial_+g \qquad (6.4.13)$$

implies [see (6.4.12) and (6.4.13)]

$$F^T=(g^T\partial_+g)^T=(\partial_+g^T)g=-g^T\partial_+g=-F \qquad (6.4.14)$$

We first investigate the arbitrariness in the solution of the first-order differential equation (6.4.12). Let g_1 and g_2 be solutions of (6.4.12) then

$$\partial_+(g_1 g_2^T) = g_1 F g_2 + g_1 F^T g_2 = \mathbb{O} \tag{6.4.15}$$

Hence

$$g_1 = A(x^-) g_2 \quad (A(x^-) \in O(N)) \tag{6.4.16}$$

The equation (6.4.12) has a particular solution $g = g_2 = B(x^+)$. Consequently the general solution of (6.4.8) is given by (6.4.11). Notice that $A = A(x^-)$ and $B = B(x^+)$ both are particular solutions of (6.4.8). Hence the left-handed and the right-handed waves pass through one another without any disturbance. This is rather similar to the free fermion theory of section 6.3. We now turn to the quantization of the W.Z.W.-model (6.4.10) in order to be able to study its equivalence with this free fermion theory.

In the light cone coordinates (6.4.7) the action (6.4.10) reads

$$S = \frac{n}{16\pi} \int d\sigma d\tau \, Tr(\partial_\tau g \partial_\sigma g^{-1}) + n\Gamma \tag{6.4.17}$$

Hence it is first order in the time derivative $\partial_\tau g$. The generalized Poisson bracket of such a Lagrangian can be determined without introducing canonical momenta (see references [13] (page 132) and [14]). For a Lagrangian of the form

$$L = \sum_{i=1}^{N} A_i(q) \dot{q}^i - V(q) \quad [q \equiv (q^1, \ldots, q^N)] \tag{6.4.18}$$

the equation of motion reads

$$\sum_{j=1}^{N} F_{ij}(q) \dot{q}_j = \frac{\partial V}{\partial q^i} \tag{6.4.19}$$

where

$$F_{ij} := \frac{\partial}{\partial q^i} A_j - \frac{\partial}{\partial q^j} A_i \tag{6.4.20}$$

Of course the equation of motion is a first-order differential equation. Let the inverse of the matrix (F_{ij}) be denoted by (F^{ij}). Then the generalized Poisson brackets are given by

$$[X(q),\ Y(q)]_{PB} = \sum_{i,j} \frac{\partial X}{\partial q^i} F^{ij} \frac{\partial Y}{\partial q^j} \tag{6.4.21}$$

Application of this expression of the Poisson bracket in the case of the W.Z.W.-Lagrangian with

$$X = Tr[M(\partial_\sigma g) g(\sigma)],\ Y = Tr[N(\partial_{\sigma'} g) g(\sigma')] \tag{6.4.22}$$

(suppressing the time τ), where M and N are $N \times N$-matrices, gives after a tedious calculation (see reference [12])

$$[X, Y]_{PB} = -\frac{4\pi}{n} \delta(\sigma - \sigma') Tr\{[M,N](\partial_\sigma g) g^{-1}\} - \frac{4\pi}{n} \delta'(\sigma - \sigma') Tr MN \tag{6.4.23}$$

Quantization is performed by the replacement of Poisson brackets by commutators of the operators corresponding to the classical quantities X and Y:

$$[\,,\,]_{PB} \rightarrow \frac{i}{\hbar}[\,,\,] \tag{6.4.24}$$

With the abbreviation

$$J_{-} := \frac{n}{2\pi}(\partial_\sigma g)g^{-1} \tag{6.4.25}$$

one gets

$$[Tr\{MJ_{-}\}, Tr\{NJ_{-}\}] = 2i\hbar\delta(\sigma-\sigma')\,Tr\{M,N]J_{-}\} + \frac{n}{\pi}i\hbar\delta'(\sigma-\sigma')\,Tr(MN) \tag{6.4.26}$$

These commutation relations coincide with those of the free fermion theory [see (6.4.24)] for

$$k = n \tag{6.4.27}$$

Finally it can be argued that the free fermion field theory with N massless Majorana fermions is equivalent with the W.Z.W.-model $n=1$ and $\lambda^2 = 4\pi$. Here only some of the arguments given in reference [12] are summarized.
Firstly with the identifications

$$J^{ij}_{-} = i\psi^i_{-}q^j_{-} = \frac{1}{2\pi}(\frac{\partial g}{\partial \sigma}g^{-1})^{ij} \tag{6.4.28}$$

and

$$J^{ij}_{+} = i\psi^i_{+}q^j_{+} = \frac{1}{2\pi}(g^{-1}\frac{\partial g}{\partial \tau})^{ij} \tag{6.4.29}$$

the currents of both theories satisfy the same commutation relations. The irreducible representation of the corresponding Kac-Moody algebras are essentially unique. Moreover the Hamiltonian H and the momentum operator **P** of both theories coincide for the case under consideration i.e. $n=1$ and $\lambda^2 = 4\pi$.
The Wess-Zumino-Witten model and its Kac-Moody algebra find a nice application in the compactification of string theory by means of group manifolds (see reference [15]).

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REFERENCES

[1] P. GODDARD and D. OLIVE, *Kac-Moody and Virasoro algebras in relation to quantum physics,* International Journal of modern physics, Vol. 1, pp. 303-414, (1986).

[2] V.G. KAC and A.K. RAINA, *Bombay lectures on highest weight representations of infinite dimensional Lie algebras,* World Scientific, Singapore New Jersey Hong Kong, 1987.

[3] H. GEORGI, *Weak interactions and modern particle theory,* The Benjamin/Cummings Publishing Company, Menlo Park, 1984.

[4] S. GASIOROWICZ, *Elementary particle physics,* John Wiley & Sons, New York London Sydney, 1966.

[5] E. WITTEN, *Global aspects of current algebras,* Nuclear Physics Vol. B223, pp. 422-432, 1983.

[6] T.H.R. SKYRME, *A unified field theory of mesons and baryons,* Nuclear Physics Vol. 31, pp 556-569, 1962.

[7] S. COLEMAN, *Quantum sine-Gordon equation as the massive Thirring model,* Physical Review D Vol. 11, pp. 2088-2097, 1975.

[8] S. MANDELSTAM, *Soliton operators for the quantized sine-Gordon equation,* Physical Review D Vol. 11, pp. 3026-3030, 1975.

[9] C.G. CALLAN, R.F. DASHEN and D.H. SHARP, *Solvable two-dimensional field theory based on currents,* Physical Review Vol. 165, pp. 1883-1886, 1968.

[10] S. COLEMAN, D. GROSS and R. JACKIW, *Fermion avatars of the Sugawara model,* Physical Review Vol. 180, pp. 1359-1366, 1969.

[11] Y.K. HA, *Boson formulation of fermion field theories,* Physical Review D Vol. 29, pp. 1744-1756, 1984.

[12] E. WITTEN, *Non-abelian bosonization in two dimensions,* Communications in Mathematical Physics Vol. 92, pp. 455-472, 1984.

[13] E.C.G. SUDARSHAN and N. MUKUNDA, *Classical dynamics: A modern perspective,* John Wiley & Sons, New York London Sydney Toronto, 1974.

[14] E.A. DE KERF and M.J. BERGVELT, *Symplectic structure and Poisson brackets* in CAP-NSERC Summer Institute in Theoretical Physics, Vol. I, pp 227-237 (F.C. KHANNA, H. UMEZAWA, G. KUNSTATTER and H.C. LEE eds.), World Scientific, 1988.

[15] D. GEPNER and E. WITTEN, *String theory on group manifolds,* Nuclear Physics B, Vol. 278, pp. 493-549, 1986.

MC SYLLABI

1.1 F. Göbel, J. van de Lune. *Leergang besliskunde, deel 1: wiskundige basiskennis*. 1965.

1.2 J. Hemelrijk, J. Kriens. *Leergang besliskunde, deel 2: kansberekening*. 1965.

1.3 J. Hemelrijk, J. Kriens. *Leergang besliskunde, deel 3: statistiek*. 1966.

1.4 G. de Leve, W. Molenaar. *Leergang besliskunde, deel 4: Markovketens en wachttijden*. 1966.

1.5 J. Kriens, G. de Leve. *Leergang besliskunde, deel 5: inleiding tot de mathematische besliskunde*. 1966.

1.6a B. Dorhout, J. Kriens. *Leergang besliskunde, deel 6a: wiskundige programmering 1*. 1968.

1.6b B. Dorhout, J. Kriens, J.Th. van Lieshout. *Leergang besliskunde, deel 6b: wiskundige programmering 2*. 1977.

1.7a G. de Leve. *Leergang besliskunde, deel 7a: dynamische programmering 1*. 1968.

1.7b G. de Leve, H.C. Tijms. *Leergang besliskunde, deel 7b: dynamische programmering 2*. 1970.

1.7c G. de Leve, H.C. Tijms. *Leergang besliskunde, deel 7c: dynamische programmering 3*. 1971.

1.8 J. Kriens, F. Göbel, W. Molenaar. *Leergang besliskunde, deel 8: minimaxmethode, netwerkplanning, simulatie*. 1968.

2.1 G.J.R. Förch, P.J. van der Houwen, R.P. van de Riet. *Colloquium stabiliteit van differentieschema's, deel 1*. 1967.

2.2 L. Dekker, T.J. Dekker, P.J. van der Houwen, M.N. Spijker. *Colloquium stabiliteit van differentieschema's, deel 2*. 1968.

3.1 H.A. Lauwerier. *Randwaardeproblemen, deel 1*. 1967.

3.2 H.A. Lauwerier. *Randwaardeproblemen, deel 2*. 1968.

3.3 H.A. Lauwerier. *Randwaardeproblemen, deel 3*. 1968.

4 H.A. Lauwerier. *Representaties van groepen*. 1968.

5 J.H. van Lint, J.J. Seidel, P.C. Baayen. *Colloquium discrete wiskunde*. 1968.

6 K.K. Koksma. *Cursus ALGOL 60*. 1969.

7.1 *Colloquium moderne rekenmachines, deel 1*. 1969.

7.2 *Colloquium moderne rekenmachines, deel 2*. 1969.

8 H. Bavinck, J. Grasman. *Relaxatietrillingen*. 1969.

9.1 T.M.T. Coolen, G.J.R. Förch, E.M. de Jager, H.G.J. Pijls. *Colloquium elliptische differentiaalvergelijkingen, deel 1*. 1970.

9.2 W.P. van den Brink, T.M.T. Coolen, B. Dijkhuis, P.P.N. de Groen, P.J. van der Houwen, E.M. de Jager, N.M. Temme, R.J. de Vogelaere. *Colloquium elliptische differentiaalvergelijkingen, deel 2*. 1970.

10 J. Fabius, W.R. van Zwet. *Grondbegrippen van de waarschijnlijkheidsrekening*. 1970.

11 H. Bart, M.A. Kaashoek, H.G.J. Pijls, W.J. de Schipper, J. de Vries. *Colloquium halfalgebra's en positieve operatoren*. 1971.

12 T.J. Dekker. *Numerieke algebra*. 1971.

13 F.E.J. Kruseman Aretz. *Programmeren voor rekenautomaten; de MC ALGOL 60 vertaler voor de EL X8*. 1971.

14 H. Bavinck, W. Gautschi, G.M. Willems. *Colloquium approximatietheorie*. 1971.

15.1 T.J. Dekker, P.W. Hemker, P.J. van der Houwen. *Colloquium stijve differentiaalvergelijkingen, deel 1*. 1972.

15.2 P.A. Beentjes, K. Dekker, H.C. Hemker, S.P.N. van Kampen, G.M. Willems. *Colloquium stijve differentiaalvergelijkingen, deel 2*. 1973.

15.3 P.A. Beentjes, K. Dekker, P.W. Hemker, M. van Veldhuizen. *Colloquium stijve differentiaalvergelijkingen, deel 3*. 1975.

16.1 L. Geurts. *Cursus programmeren, deel 1: de elementen van het programmeren*. 1973.

16.2 L. Geurts. *Cursus programmeren, deel 2: de programmeertaal ALGOL 60*. 1973.

17.1 P.S. Stobbe. *Lineaire algebra, deel 1*. 1973.

17.2 P.S. Stobbe. *Lineaire algebra, deel 2*. 1973.

17.3 N.M. Temme. *Lineaire algebra, deel 3*. 1976.

18 F. van der Blij, H. Freudenthal, J.J. de Iongh, J.J. Seidel, A. van Wijngaarden. *Een kwart eeuw wiskunde 1946-1971, syllabus van de vakantiecursus 1971*. 1973.

19 A. Hordijk, R. Potharst, J.Th. Runnenburg. *Optimaal stoppen van Markovketens*. 1973.

20 T.M.T. Coolen, P.W. Hemker, P.J. van der Houwen, E. Slagt. *ALGOL 60 procedures voor begin- en randwaardeproblemen*. 1976.

21 J.W. de Bakker (red.). *Colloquium programmacorrectheid*. 1975.

22 R. Helmers, J. Oosterhoff, F.H. Ruymgaart, M.C.A. van Zuylen. *Asymptotische methoden in de toetsingstheorie; toepassingen van naburigheid*. 1976.

23.1 J.W. de Roever (red.). *Colloquium onderwerpen uit de biomathematica, deel 1*. 1976.

23.2 J.W. de Roever (red.). *Colloquium onderwerpen uit de biomathematica, deel 2*. 1977.

24.1 P.J. van der Houwen. *Numerieke integratie van differentiaalvergelijkingen, deel 1: eenstapsmethoden*. 1974.

25 *Colloquium structuur van programmeertalen*. 1976.

26.1 N.M. Temme (ed.). *Nonlinear analysis, volume 1*. 1976.

26.2 N.M. Temme (ed.). *Nonlinear analysis, volume 2*. 1976.

27 M. Bakker, P.W. Hemker, P.J. van der Houwen, S.J. Polak, M. van Veldhuizen. *Colloquium discretiseringsmethoden*. 1976.

28 O. Diekmann, N.M. Temme (eds.). *Nonlinear diffusion problems*. 1976.

29.1 J.C.P. Bus (red.). *Colloquium numerieke programmatuur, deel 1A, deel 1B*. 1976.

29.2 H.J.J. te Riele (red.). *Colloquium numerieke programmatuur, deel 2*. 1977.

30 J. Heering, P. Klint (red.). *Colloquium programmeeromgevingen*. 1983.

31 J.H. van Lint (red.). *Inleiding in de coderingstheorie*. 1976.

32 L. Geurts (red.). *Colloquium bedrijfsystemen*. 1976.

33 P.J. van der Houwen. *Berekening van waterstanden in zeeën en rivieren*. 1977.

34 J. Hemelrijk. *Oriënterende cursus mathematische statistiek*. 1977.

35 P.J.W. ten Hagen (red.). *Colloquium computer graphics*. 1978.

36 J.M. Aarts, J. de Vries. *Colloquium topologische dynamische systemen*. 1977.

37 J.C. van Vliet (red.). *Colloquium capita datastructuren*. 1978.

38.1 T.H. Koornwinder (ed.). *Representations of locally compact groups with applications, part I*. 1979.

38.2 T.H. Koornwinder (ed.). *Representations of locally compact groups with applications, part II*. 1979.

39 O.J. Vrieze, G.L. Wanrooy. *Colloquium stochastische spelen*. 1978.

40 J. van Tiel. *Convexe analyse*. 1979.

41 H.J.J. te Riele (ed.). *Colloquium numerical treatment of integral equations*. 1979.

42 J.C. van Vliet (red.). *Colloquium capita implementatie van programmeertalen*. 1980.

43 A.M. Cohen, H.A. Wilbrink. *Eindige groepen (een inleidende cursus)*. 1980.

44 J.G. Verwer (ed.). *Colloquium numerical solution of partial differential equations*. 1980.

45 P. Klint (red.). *Colloquium hogere programmeertalen en computerarchitectuur*. 1980.

46.1 P.M.G. Apers (red.). *Colloquium databankorganisatie, deel 1*. 1981.

46.2 P.G.M. Apers (red.). *Colloquium databankorganisatie, deel 2*. 1981.

47.1 P.W. Hemker (ed.). *NUMAL, numerical procedures in ALGOL 60: general information and indices*. 1981.

47.2 P.W. Hemker (ed.). *NUMAL, numerical procedures in ALGOL 60, vol. 1: elementary procedures; vol. 2: algebraic evaluations*. 1981.

47.3 P.W. Hemker (ed.). *NUMAL, numerical procedures in ALGOL 60, vol. 3A: linear algebra, part I*. 1981.

47.4 P.W. Hemker (ed.). *NUMAL, numerical procedures in ALGOL 60, vol. 3B: linear algebra, part II*. 1981.

47.5 P.W. Hemker (ed.). *NUMAL, numerical procedures in ALGOL 60, vol. 4: analytical evaluations; vol. 5A: analytical problems, part I*. 1981.

47.6 P.W. Hemker (ed.). *NUMAL, numerical procedures in ALGOL 60, vol. 5B: analytical problems, part II*. 1981.

47.7 P.W. Hemker (ed.). *NUMAL, numerical procedures in ALGOL 60, vol. 6: special functions and constants; vol. 7: interpolation and approximation*. 1981.

48.1 P.M.B. Vitányi, J. van Leeuwen, P. van Emde Boas (red.). *Colloquium complexiteit en algoritmen, deel 1*. 1982.

48.2 P.M.B. Vitányi, J. van Leeuwen, P. van Emde Boas (red.). *Colloquium complexiteit en algoritmen, deel 2*. 1982.

49 T.H. Koornwinder (ed.). *The structure of real semisimple Lie groups*. 1982.

50 H. Nijmeijer. *Inleiding systeemtheorie*. 1982.

51 P.J. Hoogendoorn (red.). *Cursus cryptografie*. 1983.

CWI SYLLABI

1 Vacantiecursus 1984: *Hewet - plus wiskunde.* 1984.

2 E.M. de Jager, H.G.J. Pijls (eds.). *Proceedings Seminar 1981-1982. Mathematical structures in field theories.* 1984.

3 W.C.M. Kallenberg, et al. *Testing statistical hypotheses: worked solutions.* 1984.

4 J.G. Verwer (ed.). *Colloquium topics in applied numerical analysis, volume 1.* 1984.

5 J.G. Verwer (ed.). *Colloquium topics in applied numerical analysis, volume 2.* 1984.

6 P.J.M. Bongaarts, J.N. Buur, E.A. de Kerf, R. Martini, H.G.J. Pijls, J.W. de Roever. *Proceedings Seminar 1982-1983. Mathematical structures in field theories.* 1985.

7 Vacantiecursus 1985: *Variatierekening.* 1985.

8 G.M. Tuynman. *Proceedings Seminar 1983-1985. Mathematical structures in field theories, Vol.1 Geometric quantization.* 1985.

9 J. van Leeuwen, J.K. Lenstra (eds.). *Parallel computers and computations.* 1985.

10 Vacantiecursus 1986: *Matrices.* 1986.

11 P.W.H. Lemmens. *Discrete wiskunde: tellen, grafen, spelen en codes.* 1986.

12 J. van de Lune. *An introduction to Tauberian theory: from Tauber to Wiener.* 1986.

13 G.M. Tuynman, M.J. Bergvelt, A.P.E. ten Kroode. *Proceedings Seminar 1983-1985. Mathematical structures in field theories, Vol.2.* 1987.

14 Vacantiecursus 1987: *De personal computer en de wiskunde op school.* 1987.

15 Vacantiecursus 1983: *Complexe getallen.* 1987.

16 P.J.M. Bongaarts, E.A. de Kerf, P.H.M. Kersten. *Proceedings Seminar 1984-1986. Mathematical structures in field theories, Vol.1.* 1988.

17 F. den Hollander, H. Maassen (eds.). *Mark Kac seminar on probability and physics. Syllabus 1985-1987.* 1988.

18 Vacantiecursus 1988. *Differentierekening.* 1988.

19 R. de Bruin, C.G. van der Laan, J.R. Luyten, H.F. Vogt. *Publiceren met LATEX.* 1988.

20 R. van der Horst, R.D. Gill (eds.). *STATAL: statistical procedures in Algol 60, part 1.* 1988.

21 R. van der Horst, R.D. Gill (eds.). *STATAL: statistical procedures in Algol 60, part 2.* 1988.

22 R. van der Horst, R.D. Gill (eds.). *STATAL: statistical procedures in Algol 60, part 3.* 1988.

23 J. van Mill, G.Y. Nieuwland (red.). *Proceedings van het symposium wiskunde en de computer.* 1989.

24 P.W.H. Lemmens (red.). *Bewijzen in de wiskunde.* 1989.

25 Vacantiecursus 1989: *Wiskunde in de Gouden Eeuw.* 1989.

26 G.G.A. Bäuerle et al. *Proceedings Seminar 1986-1987. Mathematical structures in field theories.* 1990.

27 Vacantiecursus 1990: *Getallentheorie en haar toepassingen.* 1990.